ii

by College 1010

31 A [illegible] p. 156

Vendor gets £11,000 cash, 250 [illegible] } = £11250 = 1 say

Vendor's wife is assured for S at an ann. prem. of pS — $p = .02{,}875$

Annuity of pS $v_{38.54}$ is bought for $p.S.a$ — $a = 10.575$

$v\,\dot{a}^{1}_{38\,54}$ — $v\,\dot{a}$ — $\dot{a} = 10.492$

A charge of x per annum payable halfyearly is secured on the rev. life. int. of 38 after 54.

Total outlay = $1 + pS + apS + \dot{a}x + .08S = O$. ([illegible] $.08$)

54 dies — 38 dies

$$L \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2} \quad \tfrac{x}{2}$$

$pS \qquad pS \qquad pS$

$$\frac{x}{2} = \frac{i}{2} \times O$$

$$\frac{x-x}{2}(1+v^{1/2}) = pS$$

$$S = O\left(1+\frac{i}{2}\right)$$

$$\therefore \frac{x-x}{2}\left(1+\frac{i}{2}\right) + \frac{x-x}{2} = pS\left(1+\frac{i}{2}\right)$$

$$\text{or} \quad \frac{x-x}{2}\left(2+\frac{i}{2}\right) = pS\left(1+\frac{i}{2}\right) = p.O.(1+i)$$

$$(x-iO)\left(1+\frac{i}{4}\right) = p.O.(1+i)$$

$i = .045$

$ap = .026875$
$a(ud) = 5.75.01$
26875
1344
188
3
$ap = .28420$
$p = .02.6882$
$.31.108$
$.08$

$$\therefore 11250 + O.\left(1+\frac{i}{2}\right)(p+ap) + \dot{a}\,iO + .08 \times O\left(1+\frac{i}{2}\right) = O$$

$$\therefore O\left\{1 - \left(1+\frac{i}{2}\right)(p+ap+.08) - i\dot{a}\right\} = V$$

$$\therefore \text{outlay} = \frac{11250}{1-(1.0225)(.39108) - .49464}$$

$$= \frac{11250}{.10558} = 106{,}554 \text{ outlay}$$

Policy $= \left(1+\frac{i}{2}\right)$ outlay	=	108,951	= policy
		2,428	= premium
cost of annuity for premiums	=	30,953	
interest	=	26,349	
issue premium	=	8,716	
Price to vendor	=	11,250	
		89,196	

M A Mackenzie
Trinity College Toron

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

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INSTITUTE OF ACTUARIES.

On some Considerations suggested by the large Number of the Existing at the close of the Institute Observations of 1863. *By* Robert Ruthven Tilt, F.I.A., *of the General Life Assurance Company.*

[Read before the Institute, 28 January 1895.]

At a sessional meeting of the Institute held during the period allocated to the collection of the information required for the compilation of the New Mortality Tables, a discussion may not, I think, inappropriately be raised on various points arising in connection with the nature and adaptability for the purpose of the materials used in the construction of the principal Institute Table at present in use, and it is with this view that I venture to submit to the members of the Institute the results of some attempts which I have recently been making to trace the effects produced on the H^M Table by the fact that, at the close of the observations from which the Table is derived, there were still in existence nearly 75,000 out of 130,000 males who had come under observation; I have also endeavoured to obtain from the details given in the *Mortality Experience* published by the Institute some approximate figures as to the new business of the Twenty Offices which supplied the experience upon which the H^M Table is based.

It is now, I think, generally admitted that the rates of

mortality obtained from the experience of assured lives (such as that of the offices which contributed to the observations published in the Institute volume) depend to a great extent upon the proportion of new to old assurances contained in the exposed to risk and deaths at each age of life. In the early years of assurance, thanks to the benefit of the selection exercised by the company, the mortality is very much less than that of the general population, but the rates rapidly increase as the effect of this selection wears off, and there comes into operation the counter effect produced by the larger proportion of unhealthy lives amongst those that remain on the books than amongst those that discontinue their policies. Whilst opinions differ as to the influence and duration of these two forces, one of which diminishes and the other increases with the duration of the assurance, it is generally agreed that after the fourth or fifth year of the existence of the policy the changes in the rate due to these causes, which up to that time have been large, become gradually small.

In his paper "On the Rate of Mortality prevailing among Assured Lives as influenced by the length of time for which they have been assured" (*J.I.A.*, xv, 339), Dr. Sprague gives the following as the percentages that the actual deaths are of the "expected" (computed by the H^M Table) during the first five years of assurance for all ages at entry:

Year of Assurance	0	. .	44·6 per-cent.	
,, ,,	1	. .	71·5 ,,	
,, ,,	2	. .	88·9 ,,	
,, ,,	3 to 5	. .	98·6 ,,	*

and for the remaining years the mortality is always heavier than that given by the H^M Table. I find also in the preface to *The Institute of Actuaries Life Tables* (p. iv) it is stated that in the construction of the $H^{M(5)}$ Table the observations excluded consisted of those derived from the years of assurance 0 to 4 inclusive, and that "the particular period adopted as most suitable "generally for exclusion was determined by the marked difference "in the classified original data between the first five years of "assurance and the subsequent periods." In dealing, therefore, in the following pages with the mortality experience of assured

* Mr. Chatham points out (*J.I.A.*, xxix, 165) that this percentage should be 102·35.

lives, considered with reference to the duration of assurance and not to the age of the life, I have found it convenient to call the years of assurance 0 to 4 inclusive "years of light mortality", and the remainder of the existence of the assurances "years of heavy mortality", and the experience derived from the observations during these two periods of assurance I have called "light mortality" and "heavy mortality."

Inasmuch then as the mortality amongst assured lives should be considered both with regard to the age of the life and the age or duration of the assurance, it follows that tables prepared from statistics of assured lives, the observations being terminated at fixed dates, may differ not only from those compiled from the statistics of the general population, but also from each other, notwithstanding that the true rates of mortality as obtained by the construction from the same materials of select or analyzed tables may be found to be the same. That this difficulty in the way of obtaining a general mortality table applicable to all the uses for which it is required in life assurance business was recognized many years ago, the following extract from Mr. J. A. Higham's paper "On the value of Selection as exercised by the Policyholder" (read before the Institute 31 March 1851) (*J.I.A.*, i, 179) will show. Although the reference is to the Seventeen Offices' Experience, yet the quotation applies to any mortality table constructed from the experience of assured lives when a considerable part of the history of the policies is shut out by the close of the observations:

"Up to this point it has been assumed that the tables represent accurately the mortality which a company will have to meet. but we have now to notice that by reason of the imperfection of the data they certainly *under-rate* it. With the materials at present available one can do no more than follow the policies as long as their history is given to us. But here is the imperfection which marks all the tables, founded on office experience, which we at present possess—they treat all the policies which were existing at the period of observation as though they had lapsed. This would be of no consequence if they were of the same quality as the rest; but if (as I think is clear) they are of the *worst* quality—worse than the lapsed policies—worse than the community at large, it is evident that a Table of Expectations founded on the *complete* experience of Assurance Companies would show values very much below those here given. The measure of effect which would result from tracing all the policies to their termination may be judged from the large proportion which the existing policies bear to the whole number under observation."

It will be seen that Mr. Higham is strongly of opinion that the

lives existing at the close of the observations are of the worst quality, and by this I understand him to mean not that they are at the moment of closing the books in a worse state of health than those of similar ages and durations of assurance have been in the past, but that their future experience (that is, that excluded from observation by the close of the investigation) will be drawn almost entirely from the years of heavy mortality, whilst the experience already recorded contains a large proportion of the years of light mortality; so that, as may be gathered from the last paragraph of the quotation, the larger the proportion which the "existing" bears to the whole number under observation the greater is the proportion of years of light mortality contained in the recorded experience and the less the proportion contained in the future unrecorded years of assurance.

Applying then these doctrines to the observations collected by the Institute and published in the *Mortality Experience*, and remembering that out of 130,243 assured lives there has been a complete record of only 55,545, but that the greater part of the remaining 74,698 have contributed to the experience the light mortality existing amongst them during the early period of assurance, whilst the heavy mortality of the subsequent years has been to a great extent shut out, it is evident that the effects of this exclusion on the final H^M Table must be considerable. By availing ourselves of materials ready to hand in the *Journal* of the Institute we can without much difficulty obtain more exact information of both the quality and quantity of the experience thus shut out from observation. In vol. xx (pages 248 and 249) we find set forth full details of the well-known model office constructed by Mr. King from the data given in the Institute *Mortality Experience*. The study of his paper "On the Mortality amongst Assured Lives and the requisite reserves of Life Offices" of which this forms part is now such an integral part of the educational course which the Institute requires of its members before they can be received into the highest rank, that only a brief description of the methods and materials used in its construction is necessary. The tables of observations of Healthy Males (pages 1 to 46 of the *Mortality Experience*), from ages 18 to 67, were divided into ten groups each containing the experience of five consecutive ages at entry, thus embracing the 50 ages at which assurances are principally effected. For each group a table showing the probabilities of termination for every year of assurance was prepared, and by multiplying continuously the

total number of entrants at the ages comprised in the group by the complements of these probabilities (slightly graduated) the number of survivors at the end of each year of assurance was arrived at. Mr. King thus obtained ten different tables for the ten groups of ages, the radix of each being the number of entrants at the ages included in the groups according to the *Mortality Experience*. Just, then, as in a life table $e_x = \dfrac{l_{x+1}+l_{x+2}+\&c.}{l_x} = \Sigma_n p_x$, so here $\dfrac{l'_{x+n}}{l'_x}$ = the probability that a policy on a life aged x at entry will remain on the books at the commencement of the $n+1$st year of assurance, and e'_x is the curtate expectation of the existence of a policy as obtained by summation of all the values l'_{x+1}, l'_{x+2}, &c., and division by l'_x. On page 250, Mr. King gives a table of these "expectations of existence (policies)." It must, however, be noticed that, whilst these give the correct values of the policies of the model office, they are not quite correct for those of the Twenty Offices. Following a similar course to that which he adopted in the preparation of the Analyzed Mortality Tables, Mr. King doubled the rate of termination obtained from the observations for year "0", and assumed that the assurances in the model office all commenced at the beginning of the calendar year. On the assumption of an equal distribution of entrants throughout the year, the survivors to the end of the year "0", according to the *Mortality Experience*, had been under observation for only half-a-year since entry, and consequently Mr. King's rates of termination for the first year of assurance at age x, being denoted by q'_x, the probability of survivorship (policies) for year "0" (*i.e.*, the first six months of assurance) becomes $1-\frac{1}{2}q'_x$. Each of the values of ${}_np_x$ derived from the model office therefore requires multiplying by $\dfrac{1-\frac{1}{2}q'_x}{1-q'_x}$ in order that they may apply to the policies of the Twenty Offices, and as $e'_x = \Sigma_n{}'p_x$ the expectations of existence must be increased in the same proportion. Making this adjustment, and adding $\frac{1}{2}$ in each case to make the expectation complete, I obtain the values given in the third column of Table A. In the first column of the table is given the central age of each of the ten groups of ages at entry from which the expectations are derived: in the second column the number of entrants in each group, the entrants before age 18 being added to those of "central age at entry" 20, and the entrants after age 67 to those of central age

65. Column 4 gives the years of existence which would be due to the entrants of column 2 in a "complete experience"—that is, one in which every policy is traced to its termination either by death or discontinuance, on the assumption that the "existing at the close of the observations" were in their future experience subject to the same rates of termination as policies of the same ages and durations have been in the past. This, it may be remarked, is the hypothesis underlying the Final Series method applied by the Americans to the experience of their Thirty Offices.

Table A.

Central Age at Entry (1)	Number of Entrants (2)	Complete Expectation of Existence $(\overset{\circ}{e}'_{x+\frac{1}{2}})$ (3)	Complete Existence in Years (2)×(3) (4)
20	11,200	20·96	234,752
25	22,631	23·73	537,033
30	26,821	23·41	627,880
35	23,462	22·07	517,806
40	17,610	20·11	354,137
45	12,044	17·81	214,504
50	7,945	15·87	126,087
55	4,473	13·84	61,906
60	2,516	11·44	28,783
65	1,541	10·01	15,419
All Ages at Entry	130,243	20·87	2,718,307

It will be noticed that the mean duration of assurance for all ages at entry is 20·87 years, as against 9·14 given by the records of the Twenty Offices' experience.

The next step is to obtain the number of years of existence contributed by these entrants to the observations collected by the Institute. On page 13 of the *Mortality Experience*, it will be seen that the 130,243 healthy males lived or entered upon 1,283,034 years of life.

Making the usual adjustment for deaths and discontinuances, we find that the number of complete years of existence recorded is 1,190,140, so that the number of years excluded from the observations was (2,718,307 − 1,190,140) 1,527,167, or about 30 per-cent more than the number of years for which observations have been recorded.

It is, however, more important, in consequence of the special

influences to which the mortality amongst assured lives is subject, to determine what proportion of the experience, included and excluded, consists of mortality derived from observations recorded during the first four or five years of assurance. With this object in view, I again, making use of Mr. King's results, calculated the "temporary expectations of existence" for all ages at entry given in column 3 of Table B. The following is a brief description of the method pursued in arriving at these results. To make (approximately) the correction already alluded to, I inserted in each column of the figures on page 248 of vol. xx, between the years "0" and "1", the mean of the two numbers given against those years: thus, in the column headed 20 the number inserted was $\frac{8{,}364+8{,}882}{2}$: then, casting the results thus obtained horizontally across the columns, I obtained the total number of the entrants in year "0" of whom the survivors at the end of that year are given, by casting in a similar way the figures immediately below: continuing the process, the survivors at the end of the years of assurance 1, 2, 3 and 4 are obtained. Denoting these by s_0, s_1, s_2, s_3, s_4, and s_5,

$$_{4\frac{1}{2}}\mathring{e}' = \frac{\frac{1}{2}s_1+s_2+s_3+s_4+s_5+\frac{1}{2}(s_0-s_5)}{s_0}$$

$$_{4}\mathring{e}'_{\frac{1}{2}} = \frac{s_2+s_3+s_4+s_5+\frac{1}{2}(s_1-s_5)}{s_1},$$

the other values being obtained in a similar way.

I next extracted from the Tables of Observations of Healthy Male Lives given in the Institute *Mortality Experience* (pages 1 to 16) the numbers (for all ages at entry) of the "existing" at the end of years of assurance 0, 1, 2 and 3; they are given in the second column of Table B. Column 4 gives the number of the years of light mortality which would have been contributed after 31 December 1863 (or the proximate date when the observations were closed) to a "complete experience", and by multiplying the total number of entrants, namely, 130,243, by the first value in column 3, the number is obtained of the years of light mortality which would be included in the "complete" experience, namely, 514,720; then, by deducting from this number the 51,503 years of light mortality presumed to have been experienced after the close of the observations, the number of years of light mortality recorded in the Institute *Mortality Experience* is obtained, namely, 463,217.

Table C gives a summary of the results obtained to this point.

TABLE B.—*All Ages at Entry.*

Year of Assurance (1)	Existing at Beginning of Year (2)	Temporary Expectation of Existence $_{\overline{4\frac{1}{2}-n\vert}}\mathring{e}'_{n}$ (3)	(2)×(3) (4)	Years of Duration of Assurance (n)
0	...	3·952	...	0
1	6,272	3·507	21,996	$\frac{1}{2}$
2	5,573	2·758	15,370	$1\frac{1}{2}$
3	4,985	1·899	9,477	$2\frac{1}{2}$
4	4,780	·977	4,660	$3\frac{1}{2}$
...	21,610	...	51,503	...

TABLE C.—*All ages at entry.—Analysis of "Complete Experience."*

	Years of Heavy Mortality	Years of Light Mortality
Observations recorded . .	726,923	463,217
Observations not recorded .	1,476,164	51,503
	2,203,087	514,720

It will be seen from these figures that the number of years of heavy mortality included in the *Mortality Experience* is less than half the number excluded, whilst the number of years of light mortality included is more than nine times the number excluded: or, to put it in a perhaps clearer way, 39 per-cent of the observations recorded in the *Mortality Experience* are drawn from years of light mortality (years "0" to "4" of assurance), whilst of those not recorded 3·3 per-cent only would be derived from that source; and as a consequence in a complete experience (that is one in which every policy is traced to its termination by death or lapse) the years of light mortality would contribute 19 per-cent of the whole experience. I may, perhaps, anticipate criticism on these results by admitting that the observations excluded from the experience would not all affect the final table to the same extent. Thus in the H^M Table the rates of mortality

from ages 70 or 75 to the limiting age 102, are to a large extent derived from the experience of years of heavy mortality (the new business at these ages being of course very small), so that if it is assumed that the effects of selection cease after the fifth year of assurance, the addition of a number of observations at those ages will not affect the final table to the extent it would at the younger ages. The large number of the existing, however, in the first few years of assurance and the fact that, as I shall presently show, the increase in the number of entrants at the younger ages was during the last few years of the investigation period very large, show that in the case of the H^M Table the experience derived from the future years of existence would be distributed throughout the various ages of life, and would not be confined to any particular period.

I have already briefly alluded to the method of Final Series adopted in the preparation of the Mortality Tables derived from the Experience of the Thirty American Offices, and a more detailed reference to it will not at this point be out of place. A full account of the materials and the method of construction of the Tables is given in Mr. Meech's "System and Tables of Life Insurance." In the male section of the experience there were no less than 982,734 entrants, of whom 527,157 were existing on 31 December 1874, 411,092 had discontinued, and 44,485 only had died. The first 10 or 15 years' duration comprises the greatest part of the business, and the whole is practically limited to 30 years. The observations were terminated on 31 December 1874, and during the preceding 15 years there had been a very remarkable increase in life assurance business in the United States. We are told in the preface that the business of a group of companies, comprising the greater part of those in the Union, had grown to an enormous extent. Fifty thousand policies in 1859 increased to 100,000 in 1863, whilst at the end of 1865 the number exceeded 200,000; in 1867 it had risen to 400,000, and in 1872 to 800,000. As a result of this large influx of new business we are not surprised to find that a very large proportion of the experience is derived from the years of assurance 0 to 4 (that is the years of "light mortality"). Out of 4,327,086 years of exposure, no less than 2,818,397, or 65 per-cent, appertain to this period of duration. In the Institute Experience, out of 1,200,400 years of exposure, 464,224 or 39 per-cent belong to years 0 to 4. It is worthy of note that notwithstanding

the short average duration of the American policies during the period from which the observations were drawn, only 53·7 per-cent of the entrants are classified as existing, whilst in the British Experience (with an average duration more than double that of the American) the existing form 57·4 of the entrants. Differences in the rates of discontinuance of course account for this.

The method of Final Series consists in the application to the existing of the rates of termination by death and discontinuance which policies of similar duration and on lives of the same age have experienced in the past, thus tracing them to their termination by death or discontinuance, and these hypothetical results are then combined with the recorded facts. On pages 162 and 163 of Mr. Meech's work are given tables of the "logarithms of the multipliers" for Final Series, that is, the logarithms of the numbers by which the exposed to risk and deaths actually recorded are to be multiplied to give effect to this hypothetical treatment of the existing. As an illustration of the effect on the exposed to risk and deaths we may take the following figures which relate to age of exposure 60:

The numbers for the year of assurance	0	are	not affected	
,, ,, ,, ,, ,,	5	,,	multiplied by	$1\frac{1}{2}$
,, ,, ,, ,, ,,	10	,,	,, ,,	$6\frac{1}{2}$
,, ,, ,, ,, ,,	20	,,	,, ,,	21
,, ,, ,, ,, ,,	25	,,	,, ,,	30

The result is of course to largely increase the proportion of the exposed to risk and deaths drawn from the experience of policies of long duration. The American Tables as constructed by this method, cannot however I think be considered typical of Mortality Tables derived from a "complete" experience. The observations of the Thirty Offices are practically limited to 30 years, and the table of logarithms of multipliers does not extend beyond 28 years, so that in the above example no entrant at any age below 32 will contribute to the exposed to risk or to the deaths at age 60. Inasmuch as out of 982,734 entrants at all ages, 380,797 entered below age 32, it is evident that if the experience were made complete, the figures for age 60 would have to be considerably increased, and as the additions would be derived from years of insurance 29 and upwards, we may assume that the rate of mortality at age 60 would be increased. Similar results would hold at other ages, and it is to be noted that the older the age at exposure the greater is as a rule the

proportion of old assurances excluded whilst the rates of mortality for ages below 40 or 45 will not differ from those of a table constructed from a "complete" experience. It should be noted that the multipliers (or weights) given in the table already referred to are discounted at 4 per-cent according to the number of years of duration, but I do not understand the reason for this. It appears to me that the question of interest cannot be brought into account in calculating the rate of mortality that may be expected at a particular age amongst a body of "mixed" insured lives.

The question as to the effect of the exclusion from the observations of the future experience of the "existing" may be approached in another way which will I think yield interesting results. Turning to the Tables of Observations of Healthy Male Lives in the *Mortality Experience* published by the Institute (pages 1 to 46) it will be seen that there is given at the head of the columns age by age the total number of entrants during the whole period over which the observations extend, and that these entrants are traced year by year through their existence until withdrawal either by death or discontinuance or until the close of the observations. It may also be noticed that with respect to those of the entrants whose record was terminated either by death or discontinuance there are no means of telling at what part of the period comprised in the experience they were under observation. It may have been, so far as can be gathered from the information given in the book, at any time between 1720 and 1863. With the "existing", however, the case is different. It is known that they were on the books at the close of the observations and the number of years that they had been under observation is also known: for example, on page 18, current age at entry 30, it is stated that there were existing 319 who were then at the end of the year "0" of assurance, that is, 319 survivors from those who entered on the books of the Twenty Companies during the last year from which the observations were drawn (generally 1863). Similarly there were 162 survivors from the entrants during the tenth year preceding the close (say 1854). If therefore we knew the rates of termination to which those who entered at age 30 ten years before the close of the observations had been subject year by year of assurance, the probability of the survivorship of a policy could be found, and the multiplication of 162, the number of survivors, by the reciprocal of this probability would give the number

of entrants at age 30 in the year 1854. It is not possible, however, from the materials at our command to distinguish the rates of termination which were operating at any particular time, but by assuming that the rates for each year of assurance given by the aggregate figures of the experience for various groups of ages at entry are applicable to cases of similar ages and durations at any part of the period over which the observations extend, a very close approximation, at all events for the years immediately preceding the close of the observations, may be obtained of the new business (by lives) transacted by the Twenty Offices. With the aid of the materials for Mr. King's model office given on page 248 of his paper, I prepared tables of the "probabilities of the survivorship of policies" for quinquennial groups of ages at entry. Then collecting from the Tables of Observations of Healthy Male Lives given in the *Mortality Experience* the number of existing at each period of assurance for the various groups of ages at entry, and multiplying by the reciprocals of the corresponding probabilities of survivorship, the new business (by lives) of the Twenty Offices is obtained as given in a condensed form in the following Table D.

TABLE D.—*New Business of the Twenty Offices.*

Approximate Year of Entry	Ages at Entry				Total	Mean Multipliers or Weights for Final Series	Year of Assurance
	20 & 25	30 & 35	40 & 45	50, 55, 60, & 65			
1863	1,988	2,520	1,218	528	6,254	1	0
1862	1,842	2,376	1,260	567	6,045	1·052	1
1861	1,809	2,177	1,254	500	5,740	1·107	2
1860	1,850	2,242	1,176	537	5,805	1·165	3
1859	1,593	2,051	1,084	478	5,206	1·23	4
1858	1,423	1,848	1,024	487	4,782	1·295	5
1857	1,281	1,774	1,013	473	4,541	1·361	6
1856	1,304	1,984	1,046	518	4,852	1·431	7
1855	1,439	2,127	1,120	562	5,248	1·513	8
1854	1,478	2,179	1,187	591	5,435	1·614	9
1853	1,301	2,272	1,269	605	5,447	1·733	10
1852	1,256	1,915	1,094	574	4,839	1·871	11
1851	1,144	1,870	1,090	532	4,636	2·015	12
1850	1,037	1,693	982	514	4,226	2·174	13
1849	985	1,794	1,062	474	4,315	2·343	14
1844–48	3,840	6,606	4,017	2,008	16,471	...	...
1839–43	2,358	4,673	2,878	1,717	11,626	...	...
1838 & previously	3,585	8,182	5,880	4,356	22,003	...	...
Total . .	31,513	50,283	29,654	16,021	127,471	...	...

TABLE E.—*Contribution per-cent of various Groups of Ages at Entry to Total Entrants.*

Approximate Year of Entry	20 & 25	30 & 35	40 & 45	50, 55, 60, & 65
1863	31·8	40·3	19·4	8·5
1862	30·4	39·3	20·9	9·4
1861	31·5	38	21·9	8·6
1860	31·9	38·6	20·2	9·3
1859	30·6	39·4	20·8	9·2
1858	29·7	38·7	21·4	10·2
1857	28·2	39	22·4	10·4
1856	26·9	40·8	21·6	10·7
1855	27·4	40·6	21·3	10·7
1854	27·2	40·1	21·9	10·8
1853	23·9	41·7	23·3	11·1
1852	26	39·6	22·6	11·8
1851	24·7	40·4	23·5	11·4
1850	24·5	40·2	23·2	12·1
1849	22·8	41·5	24·6	11·1
1844–48	23·3	40·2	24·4	12·3
1839–43	20·3	40·2	24·7	14·8
1838 & previously	16·3	37·2	26·7	19·8
Whole Period of Observation	24·8	39·4	23·2	12·6

The new business is shown separately for the 15 years immediately preceding the close of the observations, and then in quinquenniums for the previous 10 years, the remainder being given in one group. It may be seen that according to these figures the entrants during the last 20 years of the investigation period formed no less than 75 per-cent of the total number, and from the following Table E that they entered on the average at considerably younger ages than in the earlier years. Both these facts operate in the same direction, to shut out from the experience a large number of years of observation, chiefly, of course, years of heavy mortality. These two tables may, I think, be of some interest, as serving to indicate the progress of life assurance business in the United Kingdom during a considerable period of time anterior to the commencement of the present system of the annual publications of returns made by the companies to the Board of Trade.

Table E shows that there was no very decided change in the proportion of entrants at those ages at which assurances are

principally effected—namely, 30 to 40, but that the percentage of entrants at the earlier periods of life gradually increased from 16 per-cent in the earlier part of the investigation period to 32 per-cent in the last year, and at the older ages (40 and upwards) there was a considerable decrease in the percentage of entrants. In fact for ages above 50 there appears to have been (by reference to Table D) an absolute falling off in the new business during the last 10 or 15 years. It is difficult to say what is the effect on the final table of this change in the average age of entry when the experience is suddenly terminated shortly after a considerable proportionate increase in the influx of new business at the younger ages, but it seems probable that the large increase during the last few years prior to 1863, in the entrants at the ages 30 and 35, will diminish the effect of the heavy mortality existing amongst the survivors to those ages of entrants in previous years, and thus result in lighter rates of mortality at those ages than in the case of a mortality table formed from an experience derived from offices whose entrants were year by year the same in number at each age of life.

It is worthy of notice that if the figures of Table D were given for all ages at entry and for all durations, the multipliers required for the application of the method of Final Series to the Institute Experience could very readily be obtained. By the principle of Final Series the whole of the entrants are to be traced to their actual or hypothetical termination by death or discontinuance. In the Institute Experience the whole of the entrants contributed to the exposed to risk and the deaths of "year of assurance 0"; but the entrants in the last year of the experience are of course not included in the observations drawn from subsequent years of assurance, so that according to the figures of Table D 127,471–6,254 or 121,217 entrants supply the materials for "year of assurance 1", and the figures of that year to apply "final series" must be multiplied by $\frac{127,471}{121,217}$. Similarly for "year of assurance 0" the multiplier is $\frac{127,471}{121,217-6,045}=\frac{127,471}{115,172}$ and so on, these values, derived from the numbers of entrants at all ages, being the mean multipliers or weights. The last column of the table gives their values up to "year of assurance 14", so that a comparison may be made with those of the American Experience.

In order to further investigate the relative effects on the H^M Table of the years of light and heavy mortality I have prepared the annexed diagram. It is formed from the rates of mortality given for "under 5 years", "5 years and upwards", and those in the "total" column on page 21 of the *Mortality Experience* published by the Institute. It will be seen that at the younger ages the H^M mortality (ungraduated) differs very little from that of the first $4\frac{1}{2}$ years of assurance: so little weight has the mortality amongst policies of five or more years' duration, that the abnormal rates shown by the $H^{M(5)}$ Table at ages 20 to 30 are hardly reflected at all in the H^M Table, and from ages 25 to 30 or 35, the rapid increase in the new business which attains its maximum at age 30 nearly counteracts the effect of the increasing number of survivors from entrants at younger ages. As we progress, however, along the ages of life, and the new business gradually falls off, it is found that the H^M gradually approaches the $H^{M(5)}$ Mortality, and that the light rates of the first few years of assurance have less and less effect until from age 50 to the end of the table every feature of the $H^{M(5)}$ Mortality is faithfully reproduced in that of the H^M.

At the younger ages of life then the rates given by a "mixed mortality" table such as the H^M will approximate to those which obtain during the first few years of assurance, but after the ages 30 to 40 the "mixed" rates rapidly approach those of "years of assurance five and upwards" in consequence of the gradual decrease in the number of entrants and the increase in the number of survivors from younger ages. These distinguishing features of a mixed mortality table are I think specially prominent in the H^M Table for two reasons.

Firstly: Because the new business of the Twenty Offices was during the few years previous to the close of the observations an increasing one.

Secondly: Because this new business was proportionately greater at the younger ages at entry.

Whilst I have called attention to the fact that these peculiarities of a mixed mortality table have special prominence in the H^M Table, I am by no means sure that this renders the table any the less useful for the purposes of valuation. It is now generally admitted that to correctly estimate the cost to a company of an assurance on a life of a certain age, whether it be at the commencement of the transaction when the premium has to be determined or whether it be at a later period when the premium

having been fixed, the reserve to provide for the excess of the company's liabilities over the value of the future receipts has to be calculated, in either case it is necessary to know what proportion of assured lives of the same age under policies of similar duration may be expected to fail in each future year of their existence, and to employ those probabilities properly discounted in the calculations. Directly a mixed mortality table is used for the purpose the mortality at any age is made to depend on the relative number entering under observation at younger ages and on the proportion in which these entrants are included amongst the observations at that age. Select or analyzed tables are, therefore, the only true criteria by which to judge with accuracy the position of an office in relation to its liabilities, but they are not, so far as I am aware, employed by any office for valuation purposes, probably because of the large amount of additional work which the use of them would involve and also, perhaps, because it has become the fashion to employ the combination of the two tables known as the H^M and $H^{M(5)}$, a method of valuation which it is known results in reserves in excess of those given by the use of the H^M Table alone, but which may or may not be sufficient in consequence of the light reserves for new policies, when the liabilities are estimated by select or analyzed policy-values. It is curious, that whilst these latter show that it is chiefly in the case of new assurances that the H^M values fall short of the true reserves required, yet it is for these cases that the H^M values are retained when the combination of tables is employed, it being considered that the excess of the H^M and $H^{M(5)}$ values, over either the H^M or the select, kept in hand on policies of longer duration, will provide for the deficiency in the reserves for new cases. This is, I think, clearly shown in the following table prepared from the figures given in Table W appended to Mr. King's paper (*J.I.A.*, xx, 273). In the first quinquennium of assurance the H^M valuation shows a deficiency in the reserves of 4,812 or nearly 17 per-cent. During the second quinquennium the deficiency is reduced by 439, so that the reserves at the end of 10 years are only $7\frac{1}{2}$ per-cent short of those required by the analyzed table. As a result of a valuation at the end of the third quinquennium, the deficiency is still further reduced by 1,557, whilst during the next five years no less than 1,971 of the deficit is wiped out, and at the end of the fifth quinquennium the H^M Reserves are in excess of the analyzed, and for the

remainder of the existence of the policies the two methods of valuation give results closely approximating to each other. The results here shown cannot, I think, be without attraction to those who advocate an allowance at a valuation for the heavy initial expenses usually incident to the issue of a policy. We have an allowance for the new business of the quinquennium immediately preceding the valuation of £4,812 on an annual new business of £127,471, and this is equal to about 18*s.* per-cent of the amount remaining in force—namely, £543,574 (see Table X of Mr. King's paper). As is seen from Table F these

TABLE F.

Liabilities under Policies existing in Model Office in each Quinquennium of Insurance 3 per-cent Valuation				H^M Reserves compared with Analyzed			H^M and $H^{M(5)}$ Reserves compared with Analyzed		
Quinquennium	Analyzed Mortality	H^M	H^M and $H^{M(5)}$	Per-centage	Deficiency	Excess	Per-centage	Deficiency	Excess
1	28,978	24,166	24,166	83·4	4,812	...	83·4	4,812	...
2	58,430	54,057	58,785	92·5	4,373	...	100·6	...	355
3	78,581	75,765	78,998	96·4	2,816	...	100·5	...	417
4	90,727	89,882	92,113	99·1	845	...	101·5	...	1,386
5	96,087	96,412	97,927	100·3	...	325	101·9	..	1,840
6	93,716	94,725	95,670	101·1	...	1,009	102·1	...	1,954
7	84,155	85,701	86,261	101·8	...	1,546	102·5	...	2,106
8	69,901	71,213	71,513	101·9	...	1,312	102·3	...	1,612
9	53,274	54,140	54,310	101·6	...	866	102	...	1,036
10	36,082	36,474	36,570	101·1	...	392	101·4	...	488

new business charges are, in the case of an H^M valuation, spread over a term of years, so that at the fifth quinquennial valuation they are found to have disappeared, and the reserves are at least equal to those of a true net premium valuation. It must be admitted that in the case of a large majority of offices the premium income derived from the new business of a quinquennium is not sufficient to provide the calls upon it for claims, expenses and reserves if the latter are calculated by analyzed or select tables, and I do not think that the deficiency will in many cases be less than the 18*s.* per-cent of the sums assured, which in the case of the model office the H^M Table makes allowance for.

In a valuation by means of the H^M and $H^{M(5)}$ Tables, the

deficiency at the end of the first quinquennium as compared with the analyzed values is of course the same as in the H^M—namely, in the case of the model office 4,812, but this not only disappears at the valuation made at the end of 10 years, but an additional reserve of 355 is provided, so that the premium income derived from policies during the quinquennium following that in which they were issued not only has to bear the strain caused by the deficiency in the contribution of reserves to claims, but in addition to the ordinary increase in policy-values it has to find the sum of £5,167, probably equal to about 10 per-cent of the non-profit office premiums which would be payable during the quinquennium in respect of this business. Remembering the heavy initial commissions which are now payable for the introduction of business, it seems probable that in the case of many companies using the H^M and $H^{M(5)}$ Tables, policies do not become self-supporting until they have survived two quinquennial valuations, but that at each valuation the new business of the preceding 10 years is a charge upon the surplus provided by policies of greater duration.

When we compare, according to age at entry, the policy-values produced by the use of the H^M Table with those resulting from analyzed or select tables, we find that there is a deficiency in the former values at all ages at entry during the first five years of assurance, but that for ages at entry up to about age 30 the H^M Table gives reserves for policies of greater duration at least equal to the analyzed. A general explanation of these results may be obtained, I think, from my diagram. To produce large policy-values a rapid increase in the rate of mortality is required, and in the case of select or analyzed values, this is produced by the gradually diminishing effect of the medical selection and the increasing effect of the selection exercised by the assured against the assurers. In the case of the H^M Table there is a somewhat similar abnormal increase in the rate between the ages 30 to 40, owing to the falling off in the amount of new business transacted age by age after age 30 is passed and to the increasing number of survivors from younger ages at entry included in the exposed to risk, and as a consequence, except for the first few years of assurance, the H^M policy-values are for young ages at entry at least equal to the analyzed or select values. When, however, we come to ages above 45, we find that the H^M mortality curve differs very little from the (comparatively speaking) dead-level of the

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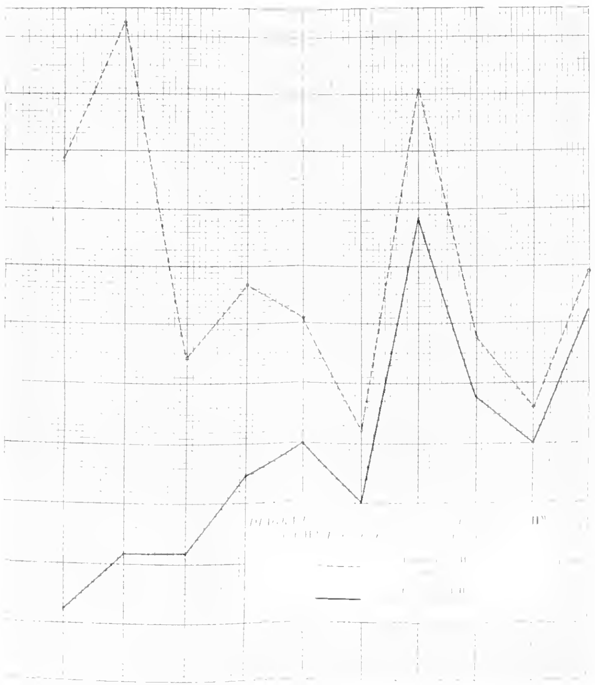


$H^{M(5)}$ Table, and as a consequence, the policy-values fall considerably short of the true values, the elasticity in the analyzed rates due to the two causes already mentioned being entirely or almost entirely lost by the preponderating influence of the mass of old assured lives.

Taking the H^M Table as a whole, Mr. King has shown that a valuation by means of it gives results less than those which would be produced by the use of analyzed or select tables, and it must therefore be concluded that a more stringent table is required to be used if results approximating to those of a true "net" premium valuation are desired. It is, however, perhaps worthy of note that the new business of the model office is assumed to be transacted at the various ages at entry in the proportions shown by the last line of the percentages given in Table E. It appears probable that the percentages given against the year 1863 in this table would more nearly represent the proportionate flow of new business into an average modern office. If this is so, then since at the younger ages at entry the H^M Reserves may be said to be equal to the select or analyzed, the difference between the reserves produced by the two methods will not be so large as is generally supposed. There can, however, be no doubt, I think, that the H^M Table does not on the whole give sufficient reserves if a true net premium valuation is desired, and if Select Tables are not to be used in a valuation, it becomes a question of the form of mixed mortality table which is best adapted for the purpose. It appears to me, from a consideration of the various points which I have mentioned, that if two tables of mortality constructed on the following lines were brought into use for the purpose of valuation, results more nearly approximating to the true reserves as given by select or analyzed tables might be the result. The first table, which should be employed only for the valuation of policies on the lives of persons whose ages at entry did not exceed 40, might be constructed from the whole experience, as is the present H^M Table, or, possibly, it might be better to shut out all entrants above age 50. The second table should be used for all the policies on the lives of persons whose ages at entry exceeded the limiting age of the first section, and would be constructed from the experience of the lives which entered at those advanced ages. In this way it appears to me probable that policy-values larger than the H^M would be obtained for policies on lives entering at the middle

ages of life, whilst the reserves for younger ages at entry would remain as at present.

To put this suggestion to a practical test would involve a large expenditure of time, but I offer it as a possible solution of what is, I think, an admitted difficulty.

DISCUSSION.

The PRESIDENT (Mr. A. J. Finlaison, C.B.) said the extent of the influence on the rate of mortality arising from the selection, exercised by assurance companies on the one hand, and by the discontinuance of assurances by healthy policyholders on the other, afforded a very suitable subject for discussion at the time when the Institute was engaged on a new observation of the rate of mortality prevailing among assured lives. He need hardly say that the Institute was endeavouring to obtain information in such a shape as to throw light upon this particular subject of enquiry, and, in this way, to aid in the solution of many of the obscurities that were still connected with the question of selection. Mr. Tilt informed them it was generally agreed that after the fourth or fifth year of the currency of policies, the changes in the rate of mortality of assured persons, due to selection of any kind, which up to that time had been large, then became gradually small. If this proposition was agreed to, the relative number of survivors after an interval of five years from the date of assurance, to the persons whose assurances terminated after a corresponding period, would have little influence on the rates of mortality deduced from an observation of the occurrences of this subsequent period. The conclusion would therefore be, that the relative number of survivors to persons whose assurances had terminated, had little influence upon the rates shown by the $H^{M(5)}$ Table where the facts of the first five years were excluded. Attention, therefore, might be restricted to the relative number of survivors who had been assured for less than five years, to those terminating within a corresponding period. In fact in the consideration of the effect of the relative number of survivors at the close of the H^{M} observations they might restrict themselves to the period which Mr. Tilt denominated the "years of light mortality." Now if the estimates of Mr. Tilt of the relative amount of facts lost to observation by the policies of survivors not being observed to their conclusion, were accepted, it would be seen from Table C that the number of years of observation not recorded in the "years of light mortality", were relatively far fewer than the number estimated not to be recorded in the "years of heavy mortality"; the comparison being made in both cases in reference to the observations that were recorded. The amount of the facts lost in the important period of the first five years was therefore, after all, relatively not so very great; in fact, only about 11 per-cent of those that were recorded. He (the President) thought that the effect of selection on the rates shown in the H^{M} Table was to be sought in the relative proportion of newly-selected lives periodically introduced into the observation rather than

in the proportion of survivors remaining during the first five years from the date of assurance. The conclusion which appeared to be forced upon them, if a low rate of mortality prevails in the first years of assurance, was that net premiums should be computed with reference to such low rate of mortality during the limited period, and that, while policies which had been in force for such short time should be valued by tables similarly constructed, those of longer duration might be valued by tables constructed in a similar manner to the $H^{M(5)}$ Table.

Mr. F. B. WYATT said the paper led to the important subject of the effect of selection, and its object seemed to be to enable them to look at that question from a new point of view. It was generally admitted that the H^M as an aggregate table was not suitable for the purposes of valuation, and a large number of offices had brought that conclusion into practical effect by using an empirical combination of the $H^{M(5)}$ and the H^M. Mr. Tilt's analysis was very opportune, seeing that they hoped before the century closed to have a collection of new facts dealing probably with nearly a million lives with a record of perhaps ten millions of years of life. When those tables came out they would probably be in a far better position to again take up the all-important question of the effect of selection, as to which opinions were, as they all knew, very different. Referring to the statement in the paper (p. 4) "that out of 130,243 assured lives there had been a complete record of only 55,545, but that the greater part of the remaining 74,698 had contributed to the experience the light mortality existing amongst them during the early period of assurance", it appeared to him that every one of the 74,000 had contributed to the experience. They had all passed through either the whole or some proportion of the first five years. The paper contained an exceedingly interesting attempt to find out the incidence of the new business of the 20 offices in each year. The conditions of business were now quite different to what they were formerly, and there were many considerations which led them to expect a change in the average age of entry. It would appear that there had been a large increase in the number of entrants at younger ages and a decrease amongst the older, and this feature would probably be even more marked in the future owing to the rapid increase of endowment assurances. Table D, giving the new business of the 20 offices, showed that in 1853 there was a sudden start in the business which remained good for 1853, 1854, and 1855, about the time of the Crimean War. It then dropped again, but in 1859—after the Indian Mutiny—the business had another accretion. Those were facts of some interest. Touching on the question of valuation, he did not think too much importance should be attached to the results deduced from a model office. Each office would be concerned with the incidence of its own business, which might differ very much from that presupposed in a model office. The author's final suggestion seemed a good one, and it had, no doubt, been suggested with a view of saving a vast amount of labour.

Mr. W. J. H. WHITTALL asked whether the author had referred to Mr. Chatham's essay (*J.I.A.*, xxix, 81), and to the various other contributions dealing with the subject under discussion, of which he would instance more particularly Mr. Sorley's valuable

paper (*J.I.A.*, xxi, 192). With regard to the final suggestion for constructing two mortality tables, of persons entering under and over a certain age respectively, he would remind Mr. Tilt that this had been already dealt with on the same basis by Mr. Ryan,* who had actually constructed two tables from the H^M data, the age of 50 being the dividing line. Coming to a general review of the paper, he regretted that the author had somewhat circumscribed his view, dealing almost entirely with the subject of the average table. What they particularly wanted more light thrown upon at the present time was not the combined effect of the two forces working in the average table, but the effect on the select tables of the force of adverse selection against the company by withdrawals. He hoped the author would see his way to investigate the subject further from that point of view. The author said (p. 3) that "tables prepared from statistics of assured lives, *the observations being terminated at fixed dates*, may differ not only from those compiled from the statistics of the general population, but also from each other, notwithstanding that the true rates of mortality as obtained by the construction from the same materials of select or analyzed tables may be found to be the same." In other words, he suggested that it was the fact of the observations being terminated at the fixed date which caused that result. "This difficulty", the author proceeded, "in the way of obtaining a general mortality table applicable to all uses for which it is required was recognized by Mr. Higham." He (Mr. Whittall) would not stop to discuss Mr. Higham's exact meaning, though it seemed to him that only the withdrawal question was involved in it; but even assuming, as Mr. Tilt had done, that Mr. Higham thought it would be desirable to replace the future experience of the existing on the basis of the past, and assuming further that it was also possible to replace this experience satisfactorily, he would like to ask what would be the result arrived at? Only an average table subject to all the defects—the well-known defects—of an average table. Thus, taking the author's illustration of two experiences where the select tables agreed, and eliminating the question of the termination at fixed dates by replacing the existing in the manner he had proposed, would Mr. Higham have solved for him the difficulty mentioned in the passage quoted, and would the average tables then be found to agree? They would not do so, simply because agreement would depend, not upon the mere eliminating of the existing element, but upon an equal distribution of new entrants; and an average table found by replacing the existing in this manner would probably, like the H^M Table itself. be still unsuitable for the calculation of premiums and for many other purposes. Passing on to Table C, which was in a sense the kernel of the paper, they would all admit that the author had rendered great service in putting the facts into such a very lucid and concrete form. But coming to the *modus operandi* by which the difficulty was to be remedied in the future the author was not quite so clear and definite. The Final Series method was the only one mentioned in that connection, and therefore it was fair to assume

* "On the several Mortality Tables employed by Life Assurance Companies in the Valuation of their Annuity Contracts", by Gerald H. Ryan, F.I.A., F.S.S. 1885. *Transactions of the Actuarial Society of Edinburgh*, New Series, No. 13.

that that was the remedy put forward to meet the case. The Final Series method had never been largely taken up in this country, and he was informed that even in its native country, America, it was not now thought much of, if it ever had been. Personally he was not at all convinced that the method was unsuitable for the purpose of improving an average table. Taking the Institute volume of *Mortality Experience*, and turning to the tables of "Exposed to Risk" which were given for every age at entry, and plotting the facts in graphic shape, it would be seen that they assumed a sort of tadpole shape. That is to say, there was a very large head representing the young business, and a very long thin tail representing the old business. Combining the overgrown heads and the attenuated tails for the purpose of forming an average table, it was obvious that the effect must be unsatisfactory, and that the table at all the earlier ages must contain an abnormal amount of this young experience. If by the Final Series method the "tails" could be so filled out that the aggregate body of the old experience was brought into better proportion with the new, it would give good *prima facie* ground for considering that the final resulting average table would be improved thereby.

After some remarks by Mr. G. F. HARDY,

Mr. W. P. PULLEY remarked that the greatest drawback to the Final Series method as hitherto set forward, in the case of American offices particularly, was the weakness of the past experience as a basis for the future, the experience already obtained being on policies of such very short duration. The rate of mortality among policies of medium duration would be rather above the average, but policies of long duration were quite as good or even better than policies taken out at the extreme ages.

Mr. C. D. HIGHAM said he did not think that Mr. J. A. Higham, whose paper had been referred to, had any idea of any method of Final Series, but he was, however, anxious to see some day a table of the experience from deaths only; and he (Mr. C. D. Higham) a few years ago tried unsuccessfully whether this could be done by a private arrangement among one or two old companies. Possibly the committee who were now preparing the new Experience might find it practicable to have such a supplementary table. It was said that dead men told no tales, but he thought such a story would be one of the most interesting they had published. He thought the use of select tables for premiums was necessary, otherwise they charged the older lives more than they ought to do to make them pay for the younger lives taken under the proper premium.

Mr. H. W. MANLY said there appeared to be two conclusions to be drawn from the paper: the first being the avoidance of the Final Series, and the second, that the H^{M} Table was a very good table to value by. He was sorry to say he could not agree with Mr. Hardy and Mr. Whittall in thinking there was something to be said in favour of Final Series. It had been assumed that withdrawals had been influenced by the state of health of the parties. Withdrawals in the early years of assurance, he thought, arose very largely from indifference. Many people were induced to insure by the sweet persuasions of the agents, and the present system of paying a procuration-fee instead of an annuity, which formerly prevailed, did

not tend to the continuance of those policies. After policies had been in force, say 10 years, he doubted very much whether the withdrawals were influenced by the state of health, as had been assumed. It was more often the state of the pocket. Pecuniary losses or diminished incomes generally compelled men to give up their assurance; and he doubted, if it were possible to trace the mortality prevailing amongst the policies that had been withdrawn, whether it would be any different from that prevailing amongst those that remained. The idea of the Final Series, as described, was to continue the existing through a longer series by using the mortality which had been found to prevail amongst a smaller number of policies taken out at the same age and in force for a longer duration. That had been described as exaggerating possibly those irregularities which arose in dealing with a small number of facts; but was the difference between the mortality amongst persons, say entering at 20, after an existence of 20 years, different from the mortality amongst those entering at 30 after an existence of 10 years? Did the effect of selection extend itself very much beyond the first five years or at all after 10 years? If not, then they had the same mortality for each attained age after 10 years. Supposing they combined the experience for all those in existence more than 10 years, then by extending the existing by such a rate of mortality they were simply multiplying ratios, and by multiplying ratios they certainly got larger figures, but they did not get different results. It appeared to him, using Mr. Whittall's illustration, that the tadpole's tail was blown out with hypothetical values which certainly did not tend to increase the value of that tail. He hoped they would have no reason, in their new Experience, to resort to any means of that kind, as he believed they would have a sufficiently large experience of the mortality at the later ages to enable them to form a complete table, and at the same time to trace the full effect of selection. The experience would be classified in such a manner that they would be able to show the course of mortality from the birth of a policy to the extreme limit of life.

The PRESIDENT, in asking the Institute to accord a vote of thanks to Mr. Tilt, said, the merit of the Final Series in its effect on a general table was that it diminished the relative effect of the light mortality among the new entrants.

Mr. TILT, in replying, said the object of the paper was not to determine the mortality table which should be used for the calculation of premiums, because no doubt select tables should and would in the future be used for such a purpose. Neither was it his object to make any fresh suggestions as to the method of construction of select tables, because with the statistics at present before them that subject had, he thought, been well threshed out. He was not sure whether he understood the President's remarks as to the light mortality not recorded being 11 per cent of that recorded. It seemed to him that the important ratios for comparison were:

$$\frac{463217}{726923} \text{ and } \frac{51503}{1476164}.$$

No doubt there was some ambiguity in the paragraph referred to by Mr. Wyatt. What he meant was that the greater part of the

74,000 had contributed the *whole* of their light mortality. It might, perhaps, be considered somewhat presumptuous to attempt an interpretation of the quotation from Mr. Higham, but he remembered that Mr. Chatham in his paper had expressed disagreement with Mr. Higham, remarking that as the existing, in being so classified, exercised no option in the matter, they could not be considered as in a worse state of health than those of similar ages and durations not so classified. It appeared to him that the meaning of the paragraph was that which was placed on it in the paper now under discussion. As there seemed to be some doubt whether he intended to advocate the adoption of the Final Series method, he wished to state that he did not consider it suitable for any use to which mortality tables were put. He did not think that it would be of any use in the calculation of premiums nor for valuation purposes. On the other hand, he considered that the peculiarities of the H^M Table were, as he stated in his paper, rather in favour of it for valuation purposes, but certainly not for the calculation of premiums. Although, as Mr. Hardy said, the American Experience seemed to require the application of the method, yet the multipliers used for the observations derived from assurances of long duration were very alarming. To have the experience derived from a small number of observations multiplied by 30 or 40, and thus made of equivalent weight to a larger number of independent observations, must produce great irregularities in the final table. To show that the effect of the method on a table such as the H^M would not be so great, he had added a column to Table D, showing the multipliers which applied to the H^M Table. In reply to the remarks of Mr. Higham and Mr. Pulley as to the mortality amongst assured lives showing improvement after the first twelve or thirteen years, the rates never became so low as in the first three or four years, and in the application of the Final Series method any such improvement would of course be reflected in the result.

I. *On the standard of Solvency in Life Assurance Companies as affected by guaranteed benefits.*

II. *On certain Methods of Reconstructing an insolvent Life Assurance Company.*

III. *Some Observations on Insurance Matters in Canada and the United States.*

By Gerald H. Ryan, F.I.A., *Actuary and Secretary of the British Empire Mutual Life Assurance Company.*

[Read before the Institute, 25 February 1895.]

I.

The standard of solvency in life assurance companies is a question of such importance that any new light, however small, that can be thrown upon it may well engage our attention.

Hence, anyone who can contribute to the illumination in the most insignificant degree may escape the reproach of having intruded himself and his views unnecessarily upon the profession. It is in this spirit that I submit the following short note to the Institute.

In June 1891, space was permitted me in the pages of the *Bankers' Magazine,* to discuss some of the popular aspects of the large subject of surrender-values. In the course of this contribution, I allowed myself to say that, as regards the guaranteeing of surrender-values, "taking a strict view of the matter, we may see "objections to mutual associations giving a guarantee of this "kind, since it implies that the reserve for a policy at valuation "should never be less than the guaranteed surrender-value; and "having no funds but those belonging to the general body of "policyholders to fall back upon, mutual societies are scarcely in "a position so to bind themselves. Pushing the argument to its "extreme conclusion, and expressing it in its most condensed form, "it may be said that they have no right to endanger the sum "assured for the sake of the surrender-value. But it is a different "thing with proprietary companies. They possess capital outside "the sum held against the assurance liability, upon which "dividends are paid out of the profits realized by the assurance "business, as a recompense for the protection of such capital; and "it is putting the latter to a most legitimate purpose to guarantee "that the surrender-values of their life policies shall not fall "below a fixed minimum."

From the criticism (both public and private) to which this article was subjected, it was made evident to me that a wide divergence of opinion existed upon the views enunciated in the paragraph just quoted. The chief questions I was led carefully to consider may be summed up as follows: (1) Does any advantage lie with proprietary offices in respect to the guarantee of surrender-values upon a fixed scale? and (2) Should a company, whether mutual or proprietary, be wound up, *as a matter of course,* in the event of its policy-reserves becoming less than the guaranteed surrender-values? It is in no controversial vein that I propose to follow up these ideas, and to ascertain whither they conduct us in the larger question, namely, when is a life office insolvent? Clearly if there is no objection to mutual societies guaranteeing surrender-values, and if as soon as their reserves become less than such values they should be wound up, we

get some indication of the conditions that may determine insolvency.

The question is a merely theoretical one in the case of the strong, well-established mutual societies of to-day, and has no practical importance; but neither in their case has the broader issue of the standard of solvency. We may therefore in our discussion of the matter fairly confine our attention to those cases where both questions become of importance. The tendency in favour of guaranteeing cash surrender-values is unmistakable; within the last few years many companies have added this new attraction to their "privileges", and who is to say that future times may not see it evolve and expand into an invariable rule with all life assurance companies. In some of our colonies, non-forfeiture regulations are compulsory by Statute: it may be that, at some time to come, their rulers will again lead the way by insisting that life offices transacting business within their limits shall guarantee fixed minimum rates for the surrender of policies. Should such a movement arise, the standard of solvency would, I shall endeavour to show, be very materially changed. I may here add that I have lately seen a statement in the public press that in Massachusetts all companies are now compelled to pay the full net reserve on surrender.

We may now turn to the writings of those who have most authoritatively dealt with the important question, when is a life office insolvent? Dr. Sprague (*J.I.A.*, xvi, 229) and Mr. Bailey (*J.I.A.*, xvi, 389) may safely be our guides in considering the matter. Between the views to which they have respectively put their names many points of agreement exist, and (so far as I have been able to see) but one main point of disagreement. They both hold that in determining whether a life office is insolvent, the best method of valuation to adopt is one that brings into account, as an asset, the gross premiums less a certain percentage for working expenses; that a true table of mortality should be used; that the rate of interest employed should be that most likely to be realized with safety in the future, and that all negative values should be excluded. The point of discord is that whereas Dr. Sprague advocates the deduction from the gross premiums of such a percentage as would fairly represent the cost at which the company *could* be worked, cancelling all commissions and other burdens not strictly necessary, Mr. Bailey maintains that the estimate of the future expenditure—that is, the deduction from gross premiums—should be equal to the average rate of the *actual*

expenditure of the insolvent company in the preceding two or three years. Incidentally, it is interesting to test these opinions by the experience we have had of insolvent companies since the time they were uttered. In no case do we find that Mr. Bailey's views as to the proper deduction to be made from the gross premiums in establishing the insolvency of an office have been acted upon, the alternative course as recommended by Dr. Sprague, in some form or other, having been invariably followed. It is easy now to examine the standard of solvency thus outlined, and to fix the limits within which it ranges. Assuming premiums to be charged of moderate character, and presenting no special feature, the criterion of solvency would probably lie between H^M $3\frac{1}{2}$ per-cent with 15 per-cent off gross premiums, and H^M 4 per-cent with 10 per-cent off—the latter the inferior and the former the superior limit of the standard—under ordinary circumstances. It will be interesting, therefore, to show the reserves that result from a gross premium valuation on these and intermediate bases, and columns may be added (to meet the object of this enquiry) showing the percentage of such reserves to the total gross premiums paid. In the following table two ages at entry have been taken, namely, 25 and 35, and the office premiums at these ages have been assumed to be as under:

Age at Entry	Profit Premiums	Non-Profit Premiums
	£ *s.* *d.*	£ *s.* *d.*
25	2 4 1	1 16 10
35	2 15 11	2 7 5

The columns headed V(1), V(2) and V(3), relate to the gross premium reserves, where 10 per-cent, $12\frac{1}{2}$ per-cent and 15 per-cent are respectively deducted for expenses and contingencies. [Formula ${}_nV_x = A_{x+n} - kP'_x\left(\frac{1}{2} + a_{x+n}\right)$, k being the percentage of gross premiums valued, *i.e.*, 90, 87·5, and 85, in the cases assumed.]

Valuations for Solvency.

1.—*Basis* H^M $3\frac{1}{2}$. *With-Profit Policies.*

Duration	Age at Entry 25—Office Premium £2. 4s. 1d.						Age at Entry 35—Office Premium £2. 15s. 11d.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	− 5·2	− 4·1	− 3·1	...	...	...	− 2·6	− 1·4	− 0·2	...	...	...
5	− 3·1	− 2·1	− 1·0	...	...	...	+ 0·4	+ 1·6	+ 2·7	3	11	19
10	+ 2·8	+ 3·8	+ 4·7	13	17	21	8·6	9·6	10·7	31	35	38
20	16·8	17·6	18·5	38	40	42	27·1	27·9	28·8	49	50	52
30	33·6	34·3	35·0	51	52	53	47·3	47·9	48·5	56	57	58

2.—*Basis* H^M $3\frac{1}{2}$. *Non-Profit Policies.*

Duration	Age at Entry 25—Office Premium £1. 16s. 10d.						Age at Entry 35—Office Premium £2. 7s. 5d.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	1·0	2·0	2·9	18	34	50	4·1	5·1	6·0	58	71	85
5	2·9	3·9	4·8	32	42	52	6·8	7·8	8·7	57	65	73
10	8·5	9·3	10·2	46	51	55	14·5	15·4	16·2	61	65	68
20	21·7	22·4	23·2	59	61	63	31·8	32·5	33·2	67	68	70
30	37·5	38·2	38·8	68	69	70	50·7	51·2	51·6	71	72	73

3.—*Basis* H^M 4. *With-Profit Policies.*

Duration	Age at Entry 25—Office Premium £2. 4s. 1d.						Age at Entry 35—Office Premium £2. 15s. 11d.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	− 6·7	− 5·8	− 4·8	...	...	...	− 4·3	− 3·2	− 2·0	...	...	...
5	− 4·5	− 3·6	− 2·6	...	...	...	− 1·4	− 0·3	+ 0·8	...	...	6
10	+ 0·8	+ 1·8	+ 2·7	4	8	12	+ 6·4	+ 7·4	8·4	23	27	30
20	14·2	15·0	15·8	32	34	36	24·7	25·5	26·3	44	45	47
30	30·9	31·5	32·2	47	48	49	44·8	45·4	46·0	53	54	55

VALUATIONS FOR SOLVENCY—(*continued*).

4.—*Basis* H^{M} 4. *Non-Profit Policies.*

Duration	Age at Entry 25—Office Premium £1. 16s. 10d.						Age at Entry 35—Office Premium £2. 7s. 5d.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	− 1·0	− 0·1	+ 0·8	...	...	15	2·0	2·9	3·8	28	41	54
5	+ 1·1	+ 2·0	2·8	12	21	30	4·7	5·6	6·4	40	47	54
10	6·2	7·0	7·8	34	38	42	12·0	12·8	13·6	51	54	57
20	18·8	19·5	20·2	51	53	55	29·2	29·9	30·5	62	63	64
30	34·6	35·1	35·6	63	64	64	48·1	48·6	49·0	68	68	69

The hope may be entertained that these figures will be useful apart from the immediate issue of this enquiry. It need hardly be said that the whole superstructure rests upon the assumed premiums, and that in the case of any company whose rates differ considerably from these, the results have no value. The larger the premium the smaller, obviously, will be the reserve, and, *a fortiori*, the smaller the percentage to total premiums paid; and *vice versâ*. An interesting point to be observed is the approximate agreement between the results of the H^{M} $3\frac{1}{2}$ with 10 per-cent thrown off the premiums, and the H^{M} 4 with 15 per-cent thrown off. This will be seen from the following excerpts from the foregoing tables.

Duration.	Age at Entry 25 (Profit)		Age at Entry 35 (Profit)		Age at Entry 25 (Non-Profit)		Age at Entry 35 (Non-Profit)	
	H^{M} $3\frac{1}{2}$ V(1)	H^{M} 4 V(3)	H^{M} $3\frac{1}{2}$ V(1)	H^{M} 4 V(3)	H^{M} $3\frac{1}{2}$ V(1)	H^{M} 4 V(3)	H^{M} $3\frac{1}{2}$ V(1)	H^{M} 4 V(3)
3	−5·2	−4·8	−2·6	−2·0	+1·0	+0·8	+4·1	+3·8
5	−3·1	2·6	+0·4	+0·8	2·9	2·8	6·8	6·4
10	+2·8	+2·7	8·6	8·4	8·5	7·8	14·5	13·6
20	16·8	15·8	27·1	26·3	21·7	20·2	31·8	30·5
30	33·6	32·2	47·3	46·0	37·5	35·6	50·7	49·0

The agreement of the H^{M} $3\frac{1}{2}$ reserves with 10 per-cent off the gross premiums, with the H^{M} 4 reserves with 15 per-cent off is, I think, sufficiently close to be worthy of remark.

Now for the purposes in hand, we may reserve our special

attention for the columns giving the several percentages of the reserves to premiums paid. In almost every case in which a company guarantees minimum cash surrender-values, such values are in proportion to the premiums paid upon the policy. The lowest percentage I have met with is 25. Let us assume a weak company to have guaranteed that at least 25 per-cent of the premiums paid shall be returned on surrender at any time after the policy has been five years in force, and see how materially this fact will influence its standard of solvency, provided there is no special fund, such as a proprietary account, to call in aid of its reserves. It would be insolvent in respect of its participating policies of short duration, even by the H^{M} $3\frac{1}{2}$ per-cent valuation with 15 per-cent thrown off, a basis on which, but for the guarantee aforesaid, many sound offices would be quite prepared to accept a transfer of the business; while in the case of valuations that sail closer to the wind the difference between the guaranteed value and the tabular reserve becomes very much wider. Bonuses may have been added to the sum assured, and a consideration of this element would complicate matters; but, both for simplicity and because the guarantee is usually irrespective of the cash value of reversionary additions we may, without serious detriment to the main argument, leave them out of the question.

Other illustrations could be taken from the preceding tables to emphasize the same point were it necessary. But the figures will, no doubt, tell their own tale patently enough. The guarantee of surrender-values upon scales at present in vogue must be admitted to involve such reserves as will greatly hasten the period when a weak life office is compelled to wind up its affairs or transfer its business to a more vigorous rival. This may be a good thing or a bad; but, beyond all reasonable difference of opinion, it is an important matter which should be carefully considered by all whom it more closely concerns. And although the drift of this short enquiry into a neglected phase of our subject is not to be considered adverse to the adoption by any substantial mutual or proprietary life office of the system of guaranteeing surrender-values, its effect should be to impress upon those other institutions which can scarcely claim to hold a position in this category, the weight of the responsibilities they assume by following the "course of trade" in this respect.

But to return to the relative positions of mutual and proprietary life offices as regards this matter. In any dispassionate

consideration of the point at issue much, it must be admitted, remains to be said on either side. A mutual office, formed to carry on the business of life assurance, must clearly be in a position to undertake such business under the conditions forced upon it by the competition of its rivals; and in this sense it is, if it should so think fit to do, just as much justified in guaranteeing surrender-values to be paid out of the common fund as in pledging its resources for any other definite purpose. Mutual companies, equally with proprietary offices, guarantee paid-up policies on a fixed scale under their endowment-assurance and limited-payment systems, and one mutual company at least extends the rule to its whole-life system. Now *in principle* it may be urged there is little to choose between such a regulation and the guarantee of fixed cash surrender-values: each course attaches to a contract a definite liability which cannot be varied in the event of the company's position becoming weaker, except as a result of a winding-up order and a general revision of contracts. No one would attempt to dispute the full right and power of mutual societies to issue policies subject to these conditions. All that need be borne in mind is that these advantages are given at the expense of and on the security solely of the general fund, with the result that, as I have endeavoured to show in the particular case of guaranteed cash surrender-values, the society may be brought to a stand-still very much sooner than would otherwise, in all probability, happen. On reflection, I am therefore disposed to modify the phrase I used— "and, having no funds but those belonging to the general body "of policyholders to fall back upon, mutual societies are scarcely "in a position to bind themselves"—but I think I am entitled to hold that there are objections to mutual associations guaranteeing surrender-values which do not apply equally to proprietary companies. And, on the ground that practice usually treads closely at the heels of theory, the fact that, out of the many British offices adopting this system as regards their ordinary contracts, two only are of mutual constitution, seems to show that the distinction is appreciated by the insurance managers of the day. Of course, as regards proprietary companies, many points have to be weighed in estimating the value of the guarantee, such as the relative magnitude of the shareholders' capital and the life assurance fund, &c.

A passing word may be devoted to the second question, namely, whether a company (mutual or proprietary) whose reserves have

fallen below the guaranteed surrender-values should be summarily wound up. Providing the surrender-values *have* been guaranteed, in the case of the mutual company such a course—or the alternative of a transfer to a solvent company—cannot, of course, be averted, since members who surrender their policies would render the company's position still weaker by withdrawing more than their strict share in the funds, to the obvious detriment of those who remained. In the case of a proprietary company, it must be a question of circumstance and of policy. The shareholding body would very likely prefer to appropriate a portion of their funds to make up the reserves to the necessary standard in the cases under consideration, rather than have their enterprise brought to ground, if, for example, the unforeseen failure of a large investment had brought about the condition of weakness with which they were called upon to deal. But as this is written with a scientific purpose and in a purely uncontroversial spirit, I will content myself with observing that companies, whether mutual or proprietary, that incorporate into their contracts a guarantee of a fixed surrender-value upon any scale now in vogue must remember that in times of emergency they may find the step has involved them in difficulties from which other companies, in even weaker financial circumstances, are free. And it could be easily shown, if it seemed necessary to do so, that the guarantee of other benefits, such as fixed paid-up policies, may tell in the same direction.

II.

A note may be added to the foregoing brief contribution with the object of bringing forward for discussion certain proposals which have been laid before the Actuarial Society of America for endeavouring to re-adjust the affairs of a life assurance company which has failed to come up to the standard of a net-premium valuation, as required by Statute in the United States. These suggestions were made by Mr. H. W. Smith, of Philadelphia, in a paper read before the Actuarial Society of America in April 1894, and a discussion thereon arose at the meeting of the Society in October 1894, in which by the courtesy of the members I was privileged to take part. The author states that "the essential " things to be observed in the attempt to put the company on its " feet are: (1) To protect the fund belonging to the policy- " holders; (2) To maintain the premium income of the company;

"(3) to give the maximum amount of insurance during the "years when it is most needed; and (4) to obtain, as far as "possible, the full recuperative power of the business."

To achieve these ends three methods of reconstruction are discussed by the author, who, however, disclaims all pretensions to originality except in the case of the third scheme. Stated in order these are as follows:

(*a*) To reduce the amount of each policy and the premiums thereon, so that the funds in hand will furnish the necessary reserve.

(*b*) To accept from each policyholder a note bearing at least 4 per-cent interest for the amount of the deficiency in the reserve.

(*c*) To reduce life policies to term policies, and to extend the term of endowment policies and the number of premiums in limited payment cases.

It is curious that the mode of reconstruction with which alone we are familiar in Great Britain, under which the amount of the policy is reduced while the old premium is maintained, should not have been even mentioned by the author. Nor can it be said that this plan is unknown to American actuaries, for a very exact description and a strong recommendation of it are given in a report of an actuarial commission, of which Mr. D. P. Fackler was a member, so long ago as October 1871.* Members of the Institute will scarcely need to be reminded that it was first brought to the notice of the profession by Dr. Sprague (*J.I.A.*, xvi, 229). Its omission is further remarkable because it complies so fully with the four conditions Mr. Smith laid down, especially in regard to the maintenance of the premium income of the company, in which respect scheme (*a*) falls short. It is, however, not necessary to dwell upon the merits of a method that has stood the test of trial and criticism so successfully. But mainly for the purpose of eliciting discussion, a few words may be said upon each of the three plans referred to by Mr. Smith.

(*a*) This has been termed the "scaling-down" process, and has simplicity to recommend it. A deficiency of, let us say, 20 per-cent in the reserve having been established, the sum assured

* *Report of the Committee on the Winding-up of Insolvent Insurance Companies* (*Convention of State Insurance Commissioners, October* 1871).

and premium under every contract will be reduced by that percentage (or a slightly larger percentage so as to create a working margin), and equilibrium in the finances thus restored (*see J.I.A.*, xx, 440). But such a plan neither "maintains the premium income of the company", nor "gives the maximum "amount of insurance during the years when it is most "needed", to quote the author's conditions. And it seems to me to have the further demerit of treating somewhat unfairly the assured under non-profit policies as compared with those under participating policies. It has, I think, been soundly ruled that participating policyholders in a company that has become insolvent must be looked upon as persons who have entered into a speculation that has turned out badly for them, and that the "over-payments" made by them with the object of securing bonuses must be regarded as forfeited or impounded (*see* Vice-Chancellor James's judgment, *J.I.A.*, xvi, 241). If we give effect to this decision, the result is to reduce participating policies to a greater extent than non-participating policies; and hence any scheme which reduces the contracts in both classes equally must be unfair to the latter. Again, the operation of the "scaling-down" process does not seem applicable to limited-payment or endowment assurances. If the premium term or the endowment term, as the case may be, had nearly expired, the all-round reduction would not be adequate for these policies. The plan here sketched out was, I understand, actually adopted in the reconstruction of the *Charter Oak Life Insurance Company*.

(*b*) The device of making up the reserves by advance-notes from the policyholders, bearing interest at 4 per-cent, is due to Mr. Sheppard Homans, who has endeavoured to apply it, I believe, to one insolvent company on whose affairs he was consulted. By this plan, the full premium income is kept up, and the interest income increased, while the sum assured is reduced by the amount of the note. The reserves being computed on a net-premium basis, a participating and non-participating policy of the same duration and age at entry would show the same deficiency and require an advance-note for an identical amount; but if the entire profits were appropriated in the first instance to the cancelment of these notes, no great injustice would be done in the long run. There would be a great objection to the method if the notes were in the nature of a personal covenant of the assured, and

not merely a hypothecation of the policies; for apart from considerations of the widely-varying value of the covenants, prudent men would hesitate to incur an obligation to set on its feet a concern that had already collapsed, whether such was due to misfortune or mismanagement.

(*c*) The third plan to be discussed is the original suggestion of Mr. Smith. He proposes that in lieu of a reduction of sum assured to make good the deficiency in the reserve, there should be an alteration in the form of the contract as regards whole-life assurances which are to be converted into long-term assurances, while limited-payment and endowment assurances are to have the premium term or the endowment term extended, until a financial equilibrium is by such means restored. Now, ingenious as this mode of reconstruction is, there are certain objections to be taken to it. One may well imagine a feeling of despair coming over policyholders of long standing when told that their contracts will only remain operative for a certain number of years, during which they will have to pay their full contract premiums. Surely they would prefer to put up with a present loss in the form of a reduction in their policies than to speculate so deeply on the uncertainty of the future, and run the risk of losing all should they survive the assigned term of years. The circumstances of most of the assured will doubtless require that their policies should represent some definite provision, and not all or nothing of their original assurance. And the system proposed is further open to criticism on the ground that the selection that could be exercised against the company would, in general language, be threefold, if not tenfold, as great as that which a reconstruction by reduction of contracts gives scope for. All lives of doubtful vitality would eagerly accept the term insurance for the full amount of their policies, and it is almost equally sure that the robust and healthy would stand aloof, and suffer their past payments to be sacrificed rather than throw good money after bad. For such reasons, I fail to perceive the merit of Mr. Smith's plan in the case of whole-life policies. But it is far more an open question whether his suggestions for dealing with limited payment or endowment assurances do not contain the germ of a really sound and desirable improvement in established procedure. The point to be aimed at in all reconstructions is, I presume, to reduce contracts as little as possible; and here we have a means of re-adjustment

by which they need not, perhaps, be reduced at all, solvency being re-established in another way. The strictly analogous method of treating whole-life policies would be, I imagine, to make the sum assured payable in full a *definite number of years after death,* the premiums being, of course, payable until the contract matures; though I do not put this forward as a practical suggestion. Of course, as regards endowment assurances, the scheme is not so easy of application as the plan hitherto adopted, though technical difficulties would never be allowed to stand in the way of an improved method of dealing with the policyholders' interests in reconstructing an insolvent company. The operation is really the reverse of the process explained by Mr. Manly (*J.I.A.*, xxvii, 362) for converting whole-life assurances into endowment assurances, or reducing the term of the latter; and many of his rules and formulas, modified as circumstances required, would be applicable. Endowment assurances of all terms, short and long, are so strikingly on the increase, that more consideration will have to be given in future to the suitability of any method of reconstruction proposed for adoption to their peculiar features and conditions than has hitherto been necessary. Whole-life assurances are, in fact, being slowly dethroned from the position of commanding supremacy which they have occupied ever since graduated premium rates were introduced, now some 130 years ago. Hence in the unfortunate contingency of any life assurance company, in time to come, having to avail itself of a scheme of revision of contracts, Mr. Smith's proposal may usefully be borne in mind.

It is not contended that the foregoing observations are more than a running commentary on the several plans discussed by Mr. Smith, or that they deal at all exhaustively with the subject matter of that gentleman's paper. Further, it should be mentioned that in criticizing the three methods of reconstruction, the standpoint has been taken of this country, where a net premium valuation is not obligatory, rather than of the United States of America, where that standard is inflexibly applied. Incidentally, I may express an opinion that the imposition of such a stringent measurement of liability upon any company for the purpose of reconstruction and for valuation after reconstruction is utterly indefensible, as it must lead to a sacrifice or loss of insurance to the policyholders which is quite unnecessary. Without re-opening the large question of the fitness of a

statutory standard of valuation, reasons may be perceived for requiring life insurance companies who are actively seeking popular support to show that they are in a position to justify the public in confiding their savings or surplus income to them.* But it must be different in the case of a company whose affairs are embarrassed and who may not wish to enlist new members. Surely in such circumstances the cancelment of the right to transact new business until its position could stand the common test would answer all essential purposes.

The late Elizur Wright stated before the National Insurance Convention of the United States, in 1871, that the net system of valuation (which was established by law in Massachusetts in 1858), was not intended to be a test of solvency. But that it is the accepted standard of solvency in some, if not all, of the States does not seem open to question; and unless the law is amended, the alternative methods of dealing with companies whose reserves have become impaired are either to employ the net system of valuation in re-adjusting the contracts or to depart from the sound principle that the same valuation by which the insolvency of an office is decreed should govern the distribution of the fund. The adoption of the latter course might lead to anomalies which have been already referred to in the *Journal* (xxii, 308).

Sufficient evidence will, however, have been given in this note of the keen and original spirit in which the problems of life assurance finance are attacked by our actuarial fraternity across the Atlantic; and it is but right to add that a glance at the publications of the Actuarial Society of America evinces a like thoroughness and resourcefulness in the treatment of questions involving the highest branches of the actuarial theory.

It may be interesting to state here that the late Elizur Wright held it to be the chief merit of a net valuation that it made it possible for a life insurance company to deal equitably with its retiring members.—"In all our programmes "you will find this statement to the policyholder that in case it becomes necessary "for him to leave the company, an equitable surrender-value will be given. " * * * * But the moment you apply a net valuation to the company, it "becomes possible to deal equitably with the retiring members, that is to say, to "allow them to go out and make only so much out of their going out as will "keep the company reasonably whole."

III.

When I was asked, a few weeks ago, to bring forward for the consideration of the members of the Institute, and for discussion at a meeting, some observations upon the points that had struck me most forcibly in insurance and actuarial circles in Canada and America, I formed the opinion that my impressions on these points would be scarcely of sufficient weight to justify a meeting of the members of the Institute being called together for the sole purpose of listening to them. I therefore strung together the short papers which have just been read, and which will, I hope, provide more solid and perhaps more important matter for discussion, and I am pleased to obey our President's behests by adding these more general remarks.

Extension of Institute Examinations.

If I were asked to mention the most striking impression left on my mind by my visit to Canada, I should say that nothing was more remarkable than the feeling of appreciation and satisfaction with which the recent extension of the examinations of our Institute is everywhere regarded. There is little doubt I think that, as time goes on, the inducements now offered to colonists to secure the diploma of our Institute will be largely taken advantage of, and the Institute has the gratification of knowing that it is not only extending its sphere of influence, but is also encouraging the study of sound principles in life assurance in a part of the world which, as everyone admits, has a great future before it, more or less remote as that future may be. There is, of course, another side to the question, and the increase in the number of our qualified Fellows at home must make everyone feel that to shut up any channel that is now open for the exercise of their talents is an unfortunate thing for them; and undoubtedly the home product of Canada will serve in time to keep out the imported article from England.

Insurance Department at Ottawa.

But little less important than what I have just said is the fact that in Ottawa the insurance department are keenly alive, as few government departments in any part of the world seem to be, to

the necessity of having trained men to deal with the special and technical subjects confided to them. Thus, in the insurance department in Ottawa, presided over by Mr. Fitzgerald, the insurance commissioner, Mr. Fitzgerald's assistant (Mr. Blackadar) is a Fellow of the Institute of Actuaries—the first colonist, I believe, to pass the whole of the examinations—and under him come two gentlemen who have passed respectively two and one of the examinations of our Institute. In fact, the principle which I gather will be acted upon in the future is to select for posts in the insurance department those persons who have resolved to qualify themselves by undergoing the tests of our Institute examinations. How great a recognition this is of our Institute it does not need me to point out. Perhaps nothing should really be regarded as a matter more worthy of gratification on the part of those who have induced the Institute to extend its boundaries in this direction. Incidentally, I may mention that the insurance commissioner has duties which are not, I daresay, within the knowledge of the profession at home. Both he and his assistants visit the head-quarters of insurance companies in Canada, and make an independent examination of the books and accounts of the company. I gather that, although they have not statutory power to sanction or withhold their sanction from any particular course which a company may have followed, anything which did not appear to the insurance commissioner to be in order would be brought to the notice of the directors of the company; and it need hardly be said that a recommendation from the commissioner would be treated with the importance which it deserved. But they also have the power of initiative to induce the government to appoint a commission to enquire into the affairs of any company.

I could not gather that there was any immediate probability of the valuation standard in Canada being strengthened. At present, the official standard is the net premium valuation on the H^M $4\frac{1}{2}$ per-cent basis. I had thought that as the more prominent companies of the United States have reduced their standard to the 4 per-cent basis, a similar course might be in prospect in Canada; but this, however, does not appear to be the case at the present time.

A Danger Incidental to Foreign Business.

A curious point may be noticed as illustrating the troubles to which companies doing foreign business may be liable. The Canadian Insurance Act of 1886 required that foreign companies should deposit with the Insurance Department a full statement of their accounts by the 1st of April in each year. This was modified by the amending Act of last July, so that the date now reads 1st of March. It is unnecessary to point out the extreme difficulty, if not impossibility, of complying with such a statutory provision. For, first of all, accounts in Canada have to be closed, returns sent to England, the English accounts closed, the whole then audited by the head office advisers, and the complete and audited returns then sent back to be filed in Ottawa. How this is to be done, under our present conditions, within the space of two months I think it would puzzle anyone to say. This individual case, however, illustrates one danger of transacting foreign business; for it must always be in the power of the local authorities, by similar provisions of an apparently innocent nature, to place such obstacles in the way of the business as to make it virtually impossible to surmount them; and, looking ahead, it is not unreasonable to say that as the local companies get stronger and larger, and are better able to meet the requirements of the people, the tendency may be to hamper foreign countries transacting business in such parts by restrictions of the character I have referred to.

Mortality in Canada.

One's impressions as to the character of risks assumed by Canadian companies are probably based on insufficient observations; but I should not hesitate to say that the habits of the people are if anything more favourable to longevity than those of residents in the United Kingdom, and there seems to be an absence of certain diseases which are very deadly at home, such, for instance, as consumption and the diseases of the respiratory organs. On the other hand, the rigours of the climate would seem to have an effect not favourable to the attainment of extreme old age, and zymotic diseases, such as typhoid, are particularly deadly. On the whole I should be inclined to think that mortality in Canada is little, if at all, inferior to that which British population exhibits. I shall not be thought going

outside the limit of subjects which interest our members if I mention incidentally a fact connected with Canadian finance which struck me as being extremely instructive.

Canadian Banking System.

It would seem that the banks in Canada are prohibited from advancing on real estate. They may take it collaterally and in supplement to other security on which advances have been made, and thus it may happen that through the failure of their other security they may be compelled eventually to rest upon the real estate; but such a state of things is extremely rare, and we may say in general terms that the banking system of Canada is cut off from all investment or speculation in real estate. Now, if we turn our eyes to Australia, and see what has lately occurred there through the banks being so heavily loaded with mortgages on freehold estate, and if we remember how greatly the difficulties of the Australian banks have been attributable to this cause, we shall see that the Canadian banks enjoy an immunity from danger in one of the most important contingencies which can affect the stability of a banking system. Throughout the whole of the recent financial crisis which, in other directions, affected Canada and the United States almost as deeply as Australia, no Canadian bank of any importance was brought to bankruptcy; and the wise heads in financial circles in Canada consider that their good fortune is very largely due to the peculiar condition of their banking system to which I have already alluded.

Interest in Canada.

You may expect me to say a few words on the subject of the rate of interest which prevails on high-class mortgages in the Dominion and as to this it appears that in Canada, more especially in the Eastern parts, as elsewhere there is a strongly marked tendency in the rate of interest to fall. In Montreal, for example, mortgages are now effected at 4½ per-cent, whereas a few years ago at least 1 per-cent higher could be obtained with ease. As we go westward we find the rate of interest higher, and in the unsettled or partially settled districts, such as the North-West and British Columbia, 6 per-cent or even 7 per-cent can at present be obtained on the best mortgages available. The security for advances in these parts cannot, however, speaking generally, be

compared with that which investors obtain say in the cities of Montreal and Toronto; and, as a matter of fact, there are very few insurance companies, either Canadian or British, who advance money on real estate. either in the North-West or British Columbia. I need hardly say that passing through the country last year there were evidences on all hands that a period of extreme depression in real estate had been in existence for some time, and had not passed away; but it would lead me too far from the object we have in view to-night to discuss the causes of this state of affairs or the outlook.

Registry of Declined Lives.

Before passing on to deal with a few points in American insurance, I would finally mention that, as regards Canada, when in Montreal, I had the opportunity of examining the system followed by all companies there of recording and exchanging information regarding declined lives. I was particularly struck with the immense body of facts which companies collect through the operation of this system. To keep the cards upon which the records are made in proper order requires, in each office, at least one clerk's sole attention: but I was informed that the value and utility of the system was not to be measured by the trouble and expense to which it put the companies. About one case a week is discovered to have been declined by some other company, although the proposal contains no intimation at all of this fact; and, whilst I have felt grave objections to adopting the system in England, I am bound to say that the unanimous opinion of American and Canadian experts is to the effect that it is a most valuable source of protection to the companies.

Actuarial Society of America.

The chief thing I have to tell the members with regard to America centres around the Actuarial Society of America, started some 10 years ago. This is now a very flourishing body. Great interest is taken in it by the leading experts and chief insurance men on the Continent, who travel great distances to be present at the sessional meetings, and it promises to develop in importance and size, and to become a worthy compeer of our Institute. A curious feature in the proceedings of the American society is, that a paper is read at one meeting and discussed at the next meeting,

held six months later. The discussion is opened by two members appointed for the purpose, whose remarks take the form of an article or printed address on the subject of the paper, which is, in fact, a closely-argued essay, and really an additional original contribution to the matter in hand. I discussed this system, which I may term a system of discussion by paper as distinct from discussion by speech, with Mr. St. John, the genial and very popular President of the American society; and he informed me the object he and his colleagues had in view in setting on foot that system was that their discussions might reach the high level attained by the discussions taking place at the Institute of Actuaries—a graceful compliment, which I am sure all members of our Institute will thoroughly appreciate, whether it be deserved or not. There is no doubt that the discussions, apart from the remarks of the two members who have consented to read the paper for the Council, are of a more set and formal character; but, as the number of actual members increases, probably this feature will dwindle in importance and pass away. There is not the least doubt that the American actuaries look to the Institute as the great home and headquarters of actuarial knowledge in the world; and although they set themselves to work out the problems which interest us according to their own individual bent, they are well acquainted with what has been written in our *Journal*, and are fully appreciative of the opinions of our experts at home.

Final Series.

At the last meeting of the Institute reference was made by Mr. Tilt, the author of the paper then considered, to the system of "final series" brought forward by Mr. Levi Meech. The preface to that important work bore the names of several well-known and influential American actuaries; but what I heard led me strongly to believe that the system of "final series" adopted by Mr. Levi Meech in tabulating his statistics did not have the full approval of the committee under whose directions the work was supposed to be issued, and I think, in future, we may take it for granted that this novel scheme rests solely on the authority of Mr. Levi Meech himself. It is right, perhaps, to mention this fact, as the plan in question has no doubt been considered by us to bear the seal and authority of the leading actuaries in America;

and greater importance has no doubt been credited to it in consequence of this assumption.

American Life Assurance Systems.

I may be excused for saying a few general words on the subject of insurance business in America; and first I would call attention to the fact that full justice is not perhaps done to insurance business as practised in America, by considering that the three companies we are aware of in Great Britain are the typical representatives or true exponents of American life assurance systems. I have no desire to utter any depreciatory words respecting our friendly rivals here; but it is a fact that there are in America many companies, of great size and undeniable repute, whose business is confined to their own country, and whose affairs are conducted in a cautious and conservative spirit and with great economy. It would be invidious for me to mention the names of the companies I have in view; but it is sufficient to say that American life assurance business cannot be properly gauged or even understood until the position and business methods of many other companies besides those with which we are so fully familiar at home are borne in mind.

High Percentage of Cancelments.

A feature of great importance in American business is the extraordinary proportion of cancelments, that is to say, lapses and surrenders, annually taking place. From a table published by one of the companies in New York, I find that, in the year 1893, the company which showed the smallest amount of cancelled policies in the year had a percentage of $6\frac{1}{2}$ to the total insurances in force at the beginning of the year. I give the figures roughly on this point, as they will be of some interest. The total insurances in force of the company in the year 1893 were nearly 200 millions of dollars. The lapsed and surrendered policies in the year amounted to nearly 13 million dollars, or $6\frac{1}{2}$ per-cent, as I have stated. Now, this is the lowest percentage shown by any company; and in the cases of other offices in America, the percentage varies from this figure up to 20 per-cent, a proportion which I venture to think is not readily conceivable in this country. The largest offices in America show a percentage of discontinued of

10 or 12 per-cent of the total amount assured in force at the beginning of the year. I imagine that in this country, even among the most progressive companies, 3 or 4 per-cent would be an unusually high rate. The table is reprinted here, as a matter of interest.

Table A.

Name of Company	Insurance in force 1 Jan. 1893	Discontinued in 1893	Per-cent
Ætna Life	$132,778,466	$10,747,087	8·1
Equitable Life, New York	850,962,245	85,211,091	10·0
Germania	65,218,895	6,432,090	9·9
Home Life	38,008,378	7,102,853	18·7
Manhattan	61,271,530	8,543,032	13·9
Massachusetts Mutual	78,467,497	8,539,873	10·9
Mutual Benefit	195,698,088	12,694,344	6·5
Mutual Life, New York	745,780,083	89,676,478	12·0
National Life, Vt.	58,678,353	6,646,181	11·3
New York Life	689,248,629	83,706,917	12·1
Northwestern	312,512,603	24,625,347	7·9
Penn Mutual	117,925,418	11,008,303	9·3
Phœnix	30,549,306	2,711,098	8·9
Provident Life and Trust	94,726,533	7,206,095	7·6
State Mutual	45,082,951	3,447,426	7·6
Union Central	64,843,548	10,375,970	16·0
Union Mutual	33,293,485	3,865,183	11·6
United States	43,728,300	8,839,254	20·2
Washington	51,561,932	7,860,882	15·2

The Discontinued Insurance includes all insurance voluntarily cancelled by the insured; but does not include Policies ceased by Death, Maturity or Expiry, nor "not taken" Policies.

Modes of Tabulating Expenditure.

The manner of showing the relative expenditure is also curious. It is customary to exhibit the total expenses as a percentage of the total income—that is, premiums and interest—and the percentage seems to vary from about 13½ per-cent to about 24 per-cent, and even higher in the case of some companies. Another system is to show the expenses as a percentage of the aggregate amount insured, which has the result of showing an extremely small fraction; but the object or advantage of it is otherwise not apparent to me.

Guarantee of Fixed Benefits.

Then again I would like to mention that the guarantee of fixed benefits is very much more common, and is carried to a very

much higher pitch in America than in this country. From a table issued by the Mutual Benefit Life Insurance Company of Newark, I see that non-forfeiture is carried through to its ultimate development; each policy has this table printed on it, which shows for each age at issue the number of years and days during which an ordinary life policy will be maintained in force after 2, 5, 10, or 15 years' premiums have been paid, and intermediate durations go in the same proportion. This table is sufficiently instructive to be reproduced, and it is given here abridged so as to show every fifth age only. I do not remember to have seen elsewhere such a complete system of non-forfeiture as is shown by the table; and I believe that other American companies are little, if anything, behind that which I have mentioned in carrying out the principle of non-forfeiture to its full development. This will probably be a surprise to many people who have come to look upon American insurance business as identified with the "tontine" element, and the total or partial sacrifice of surrender-value.

Table B.

The Mutual Benefit Life Insurance Company.

Illustrations of Non-Forfeiture System.

Showing the Term during which the full amount of the Insurance will be extended after the lapse of Ordinary Life Policies, *assuming the Premiums to have been settled on the* "All Cash" *Plan.*

Age at Issue	Number of Annual Premiums Paid before Discontinuance								Age at Issue
	2		5		10		15		
	Years	Days	Years	Days	Years	Days	Years	Days	
25	1	236	4	95	8	285	13	87	25
30	1	308	4	261	9	298	13	98	30
35	2	32	5	146	10	46	12	163	35
40	2	137	5	293	9	235	11	28	40
45	2	199	5	250	8	222	9	154	45
50	2	161	5	38	7	100	7	250	50
55	2	52	4	106	5	318	6	26	55
60	1	276	3	154	4	207	4	257	60

I gathered in discussion with a leading member of the Actuarial Society that it was quite customary for American companies to

pass their surplus through the revenue account, by including it among the premium receipts on the one hand and among the surrenders or cash bonuses on the other. The principle acted upon is that no benefit should be assumed by the company for which some corresponding contribution does not appear among the receipts. No directors ought to create liabilities—such as reversionary bonuses—by a stroke of the pen, but every transaction should be passed through revenue. This is obviously an important point to bear in mind when examining the accounts of American companies.

Nothing else occurs to me to say on the subject of insurance business in America, but anyone visiting New York cannot fail to be struck with the great beauty of the buildings in which American companies have their head-quarters. It is not usual for them to occupy the ground floor of their buildings, these being let off to banks and mercantile houses. The buildings are of huge size and sometimes of great architectural beauty. One of the finest of the new buildings that arrest the eye of any visitor to New York is that which has been put up by an assessment company which does not believe in accumulated funds; but it is no exaggeration to say that the value of their building would represent the life assurance fund of what would be considered a tolerably large assurance company in Great Britain. Another building vies with the famous Paris opera house in the richness and beauty of its hall. This is the head-quarters of the Metropolitan Life Insurance Company, an industrial company whose growth within the last few years must be considered nothing short of phenomenal, even by those who are acquainted with the wonderful results shown by our leading British industrial office. I was privileged to spend a few days as the guest of the President of one of the American universities—Cornell University—and I had the unusual experience of being lodged in the ladies' college, where more than one hundred students were in residence. I mention this visit to Cornell because I believe I nearly succeeded in inducing the learned and enterprising president, Dr. Schurman, to found a chair for the diffusion of knowledge on the theory of statistics and insurance. I have since seen an announcement that this course of study has been added to the curriculum of an institute in Vienna. But Cornell may yet be the first seat of learning in the new world to popularize this branch of knowledge. It may be said that it is unwise to encourage the study of such subjects

in view of the scanty professional prospects of a body of actuarial students. But to this I demur. Not everyone who learns boxing is destined to be a prize fighter, nor everyone who reads law to be a professional lawyer. Actuarial and statistical science would very properly have a use as part and parcel of general culture on the commercial as distinct from the classical side; nor is this all. Many of the students, quite apart from those who might find their way into insurance workshops, would doubtless be drafted into departments of business life where a sound knowledge of what is generically termed "figures" would be of the greatest advantage to them. There would be others who might become directors of companies or even Secretaries of State—say the responsible head of an insurance department itself. What a novelty, and what a gain it would be in such a case for them to have enjoyed a sound training which taught them something of the work they were called upon to direct! On the whole, therefore, I should greet with satisfaction the inclusion of our special study among the subjects to be taught in Cornell University.

After an absence of some months, many will feel that the most pleasurable event of a long tour is the return home; and, when this is achieved, the attractions of the old country stand out redoubled in size and charm. The natural beauties of the country, the tone and standard of social and commercial life, appear to be invested with qualities hitherto unobserved or but little appreciated. And in the quiet of one's fireside we can read with a new zest our "Martin Chuzzlewit" and take a delight in the keen and humorous portraiture of certain types of American citizens; while on the other hand we feel, as we must ungrudgingly admit, that as a picture of a great nation the characterization of Dickens's pages is a libel and caricature.

Discussion.

The President (Mr. A. J. Finlaison, C.B.), said it was extremely valuable to hear of the progress made by insurance offices under the restrictions imposed in America. The ultimate advantage of the system of State supervision practised in Canada and the United States still remained to be tested, and any such system would always, as Mr. Ryan indicated, be accompanied by the practical danger that it might operate to promote a monopoly of business for local institutions, or even for a favoured section of local institutions.

The papers, which the author modestly termed "short notes", opened out some of the most important considerations affecting life assurance societies. He (Mr. Finlaison) had studied the two able papers which appeared in the 16th volume of the *Journal* by Dr. Sprague and Mr. Bailey, and had been fortunate enough to have been present when they were read and discussed in 1871. Mr. Ryan referred to these authorities as safe guides on the question of the insolvency of a life office; but it should be remembered that their papers were written in 1871, and had reference to legal opinions and judicial decisions which had then been recently pronounced. In 1872, however, the Life Assurance Companies Act of that year superseded and swept away those judicial decisions and legal opinions. Whether the rules in the Act of 1872 constituted a theoretical standard of the solvency of a life assurance company might be a matter of opinion; but until they were altered they would seem to establish a legal *idea* of the standard of solvency, namely, that a company should possess funds sufficient to provide a 4 per-cent net premium reserve according to the Seventeen Offices' Experience Table. There might be grounds for the maxim that reference must be had to the gross premiums payable under policies, but, if the view he suggested was correct, the legal standard of solvency under the Act of 1872 would require a reserve in the case of every ordinary whole-life policy. It would be in the recollection of some of the members that there was an idea current in some quarters of the House of Commons in 1870 that the publication of minimum surrender-values in the answers to the Sixth Schedule of the Act of 1870 would enable policyholders always to demand as much for their policies. The endeavours that were made in 1870 to take a retrospective view of the value of an ordinary whole-life policy were likely to be repeated if any new Life Assurance Companies Bill was brought forward. Whether it was wise for life offices to guarantee the payment of some such value was not for him to say.

Mr. H. W. MANLY said as they had no *corpus vile* to be operated on, they could approach the discussion in the same spirit in which Mr. Ryan had written, namely, "with a scientific purpose, and in a purely uncontroversial spirit." The notes naturally divided themselves into two parts—first, a company must be ascertained to be insolvent before they passed to the discussion, secondly, as to how it could be reconstructed. In discussing the first question, he was inclined to draw a marked distinction between mutual and proprietary companies. With regard to the test of solvency of a mutual society, there were two stages to be considered. First, a stage at which as a going concern it could stand alone, and, secondly, a stage where it could not stand alone, but another company might be found to take over its contracts and pay them in full. The test of solvency in the first stage would be upon the plan suggested by Mr. Bailey in 1871, namely, a gross premium valuation, and a percentage, based on the actual expenses, being deducted from the value of the future premiums. With that there must be certain conditions: negative values must be omitted, for the continuance of payment of premiums was voluntary; and there should be a full reserve for all guaranteed

contracts. At that point he should look upon the company as being in the position of a trading concern which had lost its capital; but with its business intact, and having that indefinable capital called "goodwill." The second stage would be when actual insolvency had commenced, at a point when another company would be willing to take over the contracts, but where the company itself could not stand alone. There the valuation for the test of insolvency would be similar to what he had described; but instead of deducting the actual rate of expenditure from future premiums they would deduct such a rate as another company would be willing to carry on the business for. With regard to proprietary companies, the position was somewhat different—the policyholders were not co-partners. They presumably had insured in proprietary companies because they had the additional security of the capital. They had no voice whatever in the management or conduct of the business. It was sometimes considered that a proprietary company might invest in more risky securities than a mutual office because it had the capital to invest, but whoever saw a loss written off capital account? It always came off the assurance fund. When an assurance fund had been drawn upon to such a point that the company could not give a bonus or declare a profit, did they ever see the dividend on the capital cease? It often happened in some of these proprietary companies that the dividend was fixed by the deed of settlement, and never, until the company came into the Court of Chancery, did they find any capital called up to meet any deficiency. In the case of a proprietary company, before the loading upon the premiums was taken into account for valuation purposes he should require some portion of the capital to be called up. He would look upon the loading as unpaid capital, and it should follow the same course as the subscribed capital. Coming to the question of reconstruction, it was desirable to determine, first of all, what was meant by the term "reconstruction." Was it to be assumed that in dealing with a company which was too bad to stand alone they should take some measures to put it on its legs again and make it a going concern, or was it to be simply treated as a company in voluntary liquidation? If it was to be re-established as a going concern, he was not sure that scheme (*a*) "to reduce the amount of each policy and the premiums thereon", would not be the correct method to pursue. Of course, in discussing these schemes, they must remember that in America a going concern must come up to the standard of the American law, and, in that case, scheme (*a*) would be the proper method to adopt. If reconstruction was to mean simply voluntary liquidation, then they could only deal with policyholders as creditors, and the method pursued in England was, in his opinion, the only one possible. He was strongly in favour of some change in the mode of procedure. At the present time, what with motions in the Court, provisional orders, and so on, the company dragged a weary life through the Court for some two, three or four years, and the funds were enormously diminished. If any system of reconstruction was to be successful they must have some tribunal which would act expeditiously and effectually, and the tribunal he would suggest was three actuaries with full powers. He

thought that such a tribunal would have sufficient legal knowledge to do justice and equity between all parties.

Mr. A. H. Bailey said the paper took him back for a period of about a quarter of a century to events which at that time made a great sensation in the assurance world—the failure of the Albert and European Companies. One remarkable feature of those failures was that it meant the failure not of two but of fifty-seven insurance companies which had been amalgamated with those two. The complications arising were so great that in the case of the Albert a special Act of Parliament was found necessary, and the most ample powers were given to the late Lord Cairns, who was appointed arbitrator, to do what he thought best in the matter. He (Mr. Bailey) had the honour of being consulted by Lord Cairns, and discussed with him some of the subjects referred to by Mr. Ryan. Divided into two parts these are: what is the standard of solvency, and how should an insolvent company be dealt with? In the case of the two companies referred to there was no doubt as to their insolvency, because they simply could not pay their claims; but in forming an opinion upon the solvency of a life assurance company, it was essential that the value of the premiums actually payable should be estimated, and also the value of the loading which was to provide for future expenses. There was, really, not very much difference between Dr. Sprague's estimate and his own, except that he (Mr. Bailey) thought the only way to find out the cost at which a business could be carried on was to take the average cost of the last few years. It was necessary, however, to take up the part of a prophet in this matter, and to exercise the best judgment possible as to the rate of mortality founded upon actual experience, and to see, making the valuation on that basis, whether the margin was sufficient to provide expenses to carry on the future business. If a company was proved to be insolvent, the method of dealing with it was a much more difficult matter. He thought that the best way to reconstruct a company avowedly insolvent was to reduce the sums assured, the premiums remaining the same. He did not think they understood in the United States the large extent to which policies of life assurance were used for purposes of security in this country. In the case of a mortgage on a life interest, collaterally secured by a policy of assurance, he thought the mortgagee would prefer to have the principal sum reduced rather than receive a certain dividend in present cash, and the same would hold good in almost all cases where policies were used as security. With regard to the three methods suggested for reconstruction, he thought the first one would not do at all. Then again the very last thing that a policyholder would do would be to give a note of hand to a company avowedly insolvent. The notion also of reducing life policies to term policies if there was a mortgage on the life interest would be quite unsuitable. He, therefore, held rather to the opinions he had expressed formerly. He admitted the difficulty of estimating the rate of interest in the future, and in a lesser degree the rate of mortality; but they must exercise the best judgment they could form, and test the cost by the three or four previous years, and when the company was avowedly insolvent,

rather than distribute the few pounds to each claimant, he thought it was better to reduce the sum assured.

Mr. R. P. Hardy said, so far as he could remember, there had never proceeded from that Institute a sufficient condemnation of the circumstances that brought about the lamentable condition referred to in the paper, and he would therefore ask permission to say one or two words. When it was considered what assurance companies professed to be, and how far, and in most cases most truly, they posed as public benefactors, and remembering also the class of persons to whom they appealed, the most thrifty and far-seeing of their citizens, one was surprised that such organizations should ever get their affairs entangled, and still more should ever entirely break down in their engagements. He wondered how many gentlemen in that room besides Mr. Bailey and himself had any idea of what the failure of an assurance company really meant. The reports in the newspapers gave a most inadequate idea of what took place. The result was to create a perfect pandemonium: no statement which was made was believed; no explanation tendered was received. Yet there was nobody upon whose shoulders they could fix the responsibility. What was the remedy for this state of things? It was not in the multiplication of Board of Trade returns, unaccompanied by the examination of the assets. His hope rather was in the influence of that Institute, in the wider diffusion of professional knowledge and in the maintenance of that standard of conduct which it alone recognized, and to which it gave the seal of its approval. What was really required was a fundamental alteration in the law that would make every director and every actuary personally responsible in a criminal court for a fallacious or an untrue report. Dealing with Mr. Manly's suggestion, he thought there was no real difficulty in ascertaining when the affairs of a life assurance company had received such a check that its position was critical, and he would certainly say there was no doubt at all when insolvency had really begun, although he was aware that it was difficult or impossible to frame a definition that would stand legal dissection. Mr. Manly, like most of them, objected to the state of the law. The procedure must always be costly. He thought Mr. Manly was wrong in his notion as to how the capital should be used. The law as it stood, and with all submission he thought the law was right, was that the capital was only to be used when the whole of the assurance fund had been exhausted. Where the Act of 1872 was wrong, was, in not providing that when a company had reduced its contracts the interest taken by the shareholders should be entirely extinguished. No doubt, at present, the contrary was the case, and the company might get its contracts reduced and still pay dividends to its shareholders, which he submitted was a matter that required immediate alteration. With regard to the method of reconstruction to be adopted, those suggested by Mr. Smith, and quoted by the author, did not at all commend themselves to him. He thought the method adopted by Mr. Bailey and himself in the case of the Briton met the practical circumstances of the case as nearly as possible, doing something more than rough justice, namely, that practical justice, which alone was possible in the disentanglement of human affairs.

Mr. John Coles said reference had been made to the colonists passing the Institute examination, showing how closely they were united, and that if not in name they were in fact becoming one great federation. In this connection it had occurred to him that possibly in the near future those who came up for examination here might be required to have some knowledge of the laws which governed assurance business in the Colonial Empire as well as the safeguards in operation as to assurance contracts there. Speaking of the banking system in Canada, in the far west gold and silver were almost unknown, and the beaver skin was the standard of value. The modern system of banking in Canada was far in advance of that in the United States or Australia, and standing at the very entrance of this question was the clause in the Banking Act of Canada which stated that no bank might lend money on its own shares or those of any other bank, or upon mortgage of real estate or on the security of any goods, wares or merchandise, except as collateral security. That was a great hindrance to the locking up of money of banks in real estate. The paid-up capital of the Canadian banks was something like twelve millions sterling, and their liability outside that capital exceeded something like forty-four millions sterling. That was perhaps very little in excess of the capital of one huge bank in this country; but the great difference was the fact of twelve millions paid up against forty-four millions of liabilities. Comparing that with an English bank they would probably find that the corresponding paid-up capital here would not exceed two millions sterling, with possibly a reserve of one million. Taking the total assets of the Canadian banks at say sixty millions sterling, something like forty millions stood against what was called "current loans." There were no facts to guide them as to how the securities of these current loans were made up, and there might be a great deal of what was "collateral." There were, however, other safeguards to Canadian banking. In the first place, the twelve millions of capital was nearly all raised in Canada, and nearly all the directors were Canadians. It was a great element of strength that those who managed the banks managed them on the spot; and it was the common remark in Canada that it was impossible for a board sitting in London to manage either a railway or a bank with the same efficiency with which they could be managed on the spot in Canada.

Mr. W. O. Nash said on looking at the schemes suggested by the American actuary, they should not forget that they were not intended for use in a country like England, where in these matters they had the advantage of freedom from legislative restrictions. In America they appeared to be bound hand and foot to the net premium method of valuation. He had been endeavouring to find out the actual effects of the schemes described in the paper by taking age 35 at entry, and durations varying from 3 to 30 years, and assuming that the reserves were deficient one-half of the H^M 4 per-cent net reserve. It was clear that scheme (*a*) would reduce all policies and premiums one-half. According to the English method, the reduction on the sum assured would vary from about 5 per-cent up to 36 per-cent. The reduction under scheme (*b*) for the sum assured was not more than about 25 per-cent after 30 years duration, but, in addition, an

increase of payment in the form of interest was exacted. Under scheme (*c*) a policyholder after paying 20 premiums would have to put up with a term assurance for another 18 years, the full premiums being payable during that period. He hoped these figures might throw light upon the various schemes.

Mr. M. N. Adler asked if Mr. Ryan had actually heard of offices in the United States which had been successfully reconstructed either on the basis of the schemes given in the paper or on any other. He was told that there was no general bankruptcy law in the United States which held good throughout the Union. Each State had its own bankruptcy law, and it would appear that when a man failed some creditors might be favoured to the detriment of others, so that whilst A and B got their money, C and D would be left in the cold. If companies could act in that way he thought it was rather a bad look-out for British policyholders should any American company they patronized get into difficulties. He had been associated with Mr. Bailey in carrying out the reconstruction of the Great Britain Office. It was probably owing to the very strong remarks made by Sir George Jessel, then Master of the Rolls, in denouncing the attempts of wreckers to fasten upon assurance offices, that, in the case of that particular office, there was no litigation. Everything connected with the reconstruction went on smoothly; periodical valuations had since been made, and profits had been allotted to the policyholders from time to time.

The President having proposed the usual vote of thanks to the author,

Mr. Ryan, in reply, said he had a double cause for not trespassing at any great length on their attention. He had no lances to break with his friends; all he had to do was to answer one question which Mr. Adler put to him, as to whether he knew that any company in America had been successfully reconstructed upon any of the three methods referred to in the paper or upon any other method. The answer to that question should, so far as he knew, be in the negative.

CORRESPONDENCE.

ON THE VALUE OF A POSTPONED LIFE INTEREST.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—In January 1892, Dr. Sprague contributed to the *Journal* a "Case and Opinion regarding the Value of the Life Interest of a Lady in a Reversion expectant on her own death" (*J.I.A.*, xxix, 540). The point of contention was as to what part of the sum realized the lady's executors should receive in respect of the income to which she was entitled from the date of the testator's death; and

Dr. Sprague thought that the course laid down in the Opinion seemed the best that could be adopted, except that allowance should be made for the time elapsed between the death of the lady and the actual realization of the reversion, by dividing the sum ascertained as income proportionately between the two estates.

The question as to the proper sum that should be paid in respect of income postponed through the non-conversion of a reversion is, I believe, one that has not been settled entirely to the satisfaction of lawyers, and the following case, in which the point required to be dealt with in a different way from that mentioned above, may be of interest to readers of the *Journal*. In the following case, the person who was entitled, under a will, to a life interest in the proceeds of a reversion, to be converted at the testatrix's death, was not the life upon which the reversion was expectant; and the reversion did not, therefore, fall in upon the death of the lady, whose executors claimed a payment in respect of her postponed income.

Case.

Miss A, by her will, dated 4 April 1889, bequeathed the residue of her property upon trust for sale and conversion and investment of the proceeds; and to pay the income of such sale and conversion, and the actual income of her property until converted to her mother, Mrs. A, during her life.

Miss A died on 28 June 1889, leaving her mother surviving.

Mrs. A died on 20 January 1891.

Miss A's property included a vested reversionary interest in a sum of £3,000 Consols expectant on the death of a Mrs. D. This reversionary interest was not sold, but fell in by the death of this Mrs. D on 8 May 1891. Counsel was asked to advise what, if anything, the estate of Mrs. A is entitled to in respect of her daughter's reversionary interest, and how, and on what principles, the amount (if any) due to Mrs. A's estate should be ascertained, regard being had to the fact that she died before the reversion fell into possession on the death of Mrs. D.

The above case was first submitted to Counsel, who gave the following opinion:—

" Upon the principle laid down in the case of Howe *v*. Dartmouth
" (7 Vesey 137) Mrs. A was entitled to have her daughter's reversionary
" interest converted, and the income of its conversion paid to her
" during her life. As this was not done, Mrs. A's estate, now that
" the reversionary interest has fallen in, is entitled to have made
" good to it the amount which would have been paid to Mrs. A if
" the reversionary interest had been sold. The mode of ascertaining
" this amount is prescribed by such cases as the Earl of Chesterfield's
" Trusts (24 Ch. Div. 643) and *re* Hobson (34 W. R. 70). As
" Mrs. A died before the reversion fell into possession, her estate is
" entitled, when the amount of the income shall have been ascertained

"in the manner mentioned in those two cases, to such proportion of "that amount as the time she lived after Miss A's death bears to "the time which elapsed between Miss A's death and the time when "the reversion *fell into possession.*"

An Action was subsequently instituted, the decision of the Court being as follows:

The Court agreed with the above opinion, in so far as it decided that a sum ought to be paid to Mrs. A's executors, and that the principle on which this sum ought to be ascertained was that laid down in the cases to which Counsel referred; but special reference was made to Mr. Justice Kay's judgment in *re* Hobson "at top of 72nd page of 34 W. R."; and the Order went on to say that "the "sum to be paid ought to be ascertained by calculating what sum, "put out at 4 per-cent per annum on the day of the death of "Miss A, namely, 28 June 1889, and accumulated at compound "interest at the same rate, with yearly rests, and deducting income "tax, would, with the accumulations of interest, amount on the day "of the death of Mrs. A to the *Value of the Reversionary Interest* "on such day, and that the difference between these two sums "represents the income which is payable to the executors of "Mrs. A."

An important difference between Counsel's Opinion and the Court Order will be at once remarked. While Mrs. A's estate would, according to the former, benefit by the reversion falling in soon after her death, the latter does not allow this, but bases the calculation upon the value of the reversion at Mrs. A's death; the latter course being more in accordance with Mr. Justice Kay's judgment in *re* Hobson, to which special attention was directed in the Court Order. In this case, *re* Hobson, a testator by will bequeathed personal estate upon trust after payment of debts, &c., to lay out and pay income to plaintiff for life with remainders. Part of the residuary estate consisted of a contingent reversionary interest in certain funds. Testator died in 1832; and the reversion first became saleable in 1846, but it had never been sold. Since the latter date the reversion had increased enormously in value. Upon an application in effect to have the value of the reversion apportioned as between life tenant and remainder-men, Mr. Justice Kay decided that, as both parties had been interested in keeping the reversion unsold, it must be assumed to be sold at an agreed price and apportioned in the manner above stated. He decided also that there was no difference between the realization of a reversion by sale and its falling in; and that contingent and vested reversions must be treated in the same way, for the latter might be dependent upon several lives and have practically no value at first.

It is, however, worth mentioning that, in the case of an absolute reversion dependent on one life, the sum apportioned to the life tenant in the above manner will be less than the accumulations at compound interest of the sum the reversion would have realized at testator's death, had such sum been then invested at 4 per-cent. The value of an ordinary absolute reversion increases annually at a considerably less rate than 4 per-cent; in the present case, for example, the increase

in value was only about $1\frac{3}{4}$ per-cent. Taking into account, however, the various contingencies that may arise, if the reversion be not converted at the first, the course recommended by the Court seems the most equitable that could be adopted.

I am, Sir,

Your obedient Servant,

J. R. HART, F.I.A.

26 St. Andrew Square,
Edinburgh,
8 March 1895.

MR. RYAN'S PAPER ON VALUATIONS FOR SOLVENCY.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—In the hope that they may be of interest, I send herewith tables for endowment assurances payable at age 60, corresponding to those for the whole term of life given in Mr. Ryan's paper recently read before the Institute.

In Mr. Ryan's tables, as well as in my own, I am somewhat surprised to find that the percentage of the reserve values to the premiums paid in certain cases first of all diminishes with the increased duration of the policy and then increases again. The point seems to me worthy of further investigation, and I find that a very simple formula will give the value of n when the percentages cease to diminish or commence to increase.

We have $$\frac{{}_n\mathrm{V}_x}{n\mathrm{P}_x} > = < \frac{{}_{n+1}\mathrm{V}_x}{(n+1)\mathrm{P}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

according as $(n+1){}_n\mathrm{V}_x > = < n\,{}_{n+1}\mathrm{V}_x$

according as $$\frac{{}_n\mathrm{V}_x}{{}_{n+1}\mathrm{V}_x - {}_n\mathrm{V}_x} > = < n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It should be noted that the tables in Mr. Ryan's paper are concerned with the ratios of the successive terms in the two following series:

$$n\mathrm{P}_x,\ (n+1)\mathrm{P}_x,\ (n+2)\mathrm{P}_x,\ \&c.$$

$${}_n\mathrm{V}_x,\ {}_{n+1}\mathrm{V}_x,\ {}_{n+2}\mathrm{V}_x,\ \&c.$$

The first of these two series is in arithmetical progression, and its differences are a constant quantity $(=\mathrm{P}_x)$.

The differences in the second series are, however, the results of elements progressing at varying force in different directions, as is evident from the equation

$$({}_n\mathrm{V}_x+\mathrm{P}_x)(1+i)-q_{x+n}(1-{}_{n+1}\mathrm{V}_x)={}_{n+1}\mathrm{V}_x,$$

whence

$$({}_{n+1}\mathrm{V}_x-{}_n\mathrm{V}_x)=\Delta_n\mathrm{V}_x={}_n\mathrm{V}_x(i)+\mathrm{P}_x(1+i)-q_{x+n}(1-{}_{n+1}\mathrm{V}_x),$$

where, for all the cases in the tables under consideration,

$$\left.\begin{array}{ll} {}_n\mathrm{V}_x i & \text{increases} \\ \mathrm{P}_x(1+i) & \text{is constant} \\ (1-{}_{n+1}\mathrm{V}_x) & \text{decreases} \\ q_{x+n} & \text{increases} \end{array}\right\} \text{ as } n \text{ increases.}$$

The net effect of these varying elements in the case of whole term and endowment assurances is to cause the series ${}_n\mathrm{V}_x$, in most tables to increase as n increases, but not at an even rate; and the value of the above expression in (2) must depend on the rates of interest and of mortality used in the calculations.

Your obedient servant,

ERNEST WOODS.

28 *King Street,*
Covent Garden, W.C.
30 *March* 1895.

Valuations for Solvency.

Endowment Assurances payable at 60 or Death.

$$\mathrm{A}_{x+n:\overline{60-(x+n)}}-k\mathrm{P}'_{x:\overline{60-x}}\left(\tfrac{1}{2}+a_{x+n:\overline{60-(x+n)}-\frac{1}{2}}\right)$$

1.—*Basis* H$^{\mathrm{M}}$ 3½. *With-Profit Policies.*

Duration	Age at Entry 25—Office Premium £2. 15*s*. 6*d*.						Age at Entry 35—Office Premium £4. 3*s*. 0*d*.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	− 1·9	− ·7	+ ·5	..	...	6	1·3	2·8	4·2	11	22	34
5	+ 1·7	+ 2·8	4·0	12	20	29	7·2	8·5	9·9	35	41	48
10	11·5	12·6	13·6	42	45	49	24·0	25·1	26·2	58	61	63
20	37·1	37·8	38·5	67	68	69	68·6	69·1	69·5	83	83	84
30	74·0	74·3	74·6	89	89	90	...	...	...	...	...	...

Valuations for Solvency—(*continued*).

Endowment Assurances payable at 60 or Death—(continued).

2.—*Basis* H^M $3\frac{1}{2}$. *Non-Profit Policies.*

Duration	Age at Entry 25—Office Premium £2. 8s. 10*d*.						Age at Entry 35—Office Premium £3. 13*s* 0*d*.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	3·3	4·3	5·3	44	59	73	7·5	8·7	10·0	68	80	91
5	6·6	7·6	8·6	54	62	71	13·0	14·2	15·4	71	78	84
10	16·0	16·9	17·8	65	69	73	28·8	29·7	30·7	79	82	84
20	40·2	40·9	41·5	82	84	85	70·6	71·0	71·4	97	97	98
30	75·3	75·6	75·8	103	103	104	...	...	...	...	...	...

3.—*Basis* H^M 4. *With-Profit Policies.*

Duration	Age at Entry 25—Office Premium £2. 15*s*. 6*d*.						Age at Entry 35—Office Premium £4. 3*s*. 0*d*.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	– 4·0	– 2·8	– 1·7	...	...	...	– ·9	+ ·5	+ 1·8	...	4	15
5	– ·6	+ ·5	+ 1·6	...	3	11	+ 4·8	6·1	7·4	23	29	36
10	+ 8·8	9·8	10·8	32	35	39	21·4	22·5	23·5	52	54	57
20	34·0	34·8	35·5	61	63	64	66·9	67·3	67·8	81	81	82
30	72·2	72·5	72·8	87	87	87	...	...	...	...	...	...

4.—*Basis* H^M 4. *Non-Profit Policies.*

Duration	Age at Entry 25—Office Premium £2. 8*s*. 10*d*.						Age at Entry 35—Office Premium £3. 13*s*. 0*d*.					
	V(1)	V(2)	V(3)	Percentage to Prems. paid of			V(1)	V(2)	V(3)	Percentage to Prems. paid of		
				V(1)	V(2)	V(3)				V(1)	V(2)	V(3)
3	·9	1·9	2·9	12	25	39	5·0	6·2	7·4	46	57	68
5	4·1	5·0	6·0	33	41	49	10·4	11·5	12·7	57	63	69
10	13·1	13·9	14·8	53	57	61	26·0	26·9	27·9	71	74	76
20	37·1	37·7	38·4	76	77	79	68·8	69·2	69·6	94	95	95
30	73·5	73·8	74·0	100	101	101	...	...	...	...	...	...

* *Assurance Legislation in the South African Republic.*

FOR THE REGULATING OF THE BUSINESS OF THE ASSURANCE COMPANIES IN THE SOUTH AFRICAN REPUBLIC.

(LAW No. 12, 1892.)

Approved by a Resolution of the Hon. Second Volksraad, Article 654, *dated* 7 *July* 1892, *and by the Hon. First Volksraad, under Article* 889, *dated* 22 *July* 1892.

SO many British life offices now transact business in the Transvaal, or South African Republic, that both on this account and in order to make the list of Insurance Acts printed in the *Journal* complete, we have thought it desirable to give a transcript of the Law regulating Assurance Companies in the South African Republic.† This Act came into force on 1 January 1893 (clause 18), and requires every company whose head office is outside this State

1. If doing life business, to deposit £10,000 sterling in cash, bearing interest at 3 per-cent, or approved securities, with the Treasurer-General (clauses 1, 2, and 3).
2. If doing fire or accident business but not life business, to deposit £5,000 sterling or in approved securities (clauses 1, 2, and 3).
3. To take out an annual license at a cost of £20 (clause 5).
4. To file returns relating to the business transacted in the Republic, under penalties, within three months of the date to which the figures are carried (clauses 7 and 11).
5. To appoint an official to sue and be sued in the name of the company (clauses 8 and 9).

In addition to the above provisions, minor matters are dealt with and the machinery of the Act explained, in regard to which we may refer our readers to the text.

ED. *J.I.A.*

* Reprinted from Acton's Translation of the *Government Gazette*, South African Republic; 4 *August* 1892.

† "The Cape of Good Hope Life Assurance Act, 1891", and the Government Notice of 14 December 1891 relating thereto, were given in the *Journal*, xxx, 244 and 251 respectively.

Seeing that it is necessary and advisable to regulate by law the manner and conditions under which assurance companies shall be admitted to transact business in this State, it is hereby enacted as follows:

1. Each insurance company whose head office or place of business is established outside this State, and each company established in this State, whether already established or transacting business or hereafter to be established and to begin business within this State, which will be managed or represented by directors, a secretary, manager or agent, and which has been transacting business before this law was enacted, or hereafter shall begin to transact business within this State as a fire, accident, or life assurance company shall, if after enacting of this law, proceed with or begin such business in this State, be bound within three months after such a proceeding or beginning with such business to deposit as security with the Treasurer-General of this State, either in cash or other sufficient securities approved by the Treasurer-General, the Auditor-General and Registrar of Deeds, the amount of five thousand pounds sterling for fire or accident assurance, or both together, and the amount of ten thousand pounds sterling for life assurance, whether combined or not combined with fire or accident assurance, or both.

2. Each such company may, after due notice to the Treasurer-General and Registrar of Deeds, with approval of those officials, substitute any of the deposited securities for others of like value, or for cash, and such substituted securities shall in every case be dealt with as if they were originally deposited.

3. Each company that thus deposits the aforesaid sums in cash as security will be entitled to receive interest from the Government at the rate of 3 per-cent per annum on the sums deposited, payable at the end of each year.

4. For the securities named in Article 1, the Treasurer-General shall give a receipt according to the forms attached marked A and B.

5. Each such company shall be bound, before transacting any business in the State, to take out a yearly license of £20, and no such license shall be issued before proof is given to the officer issuing such licenses that the security mentioned in Article 1 of this law has been deposited.

6. Each such company shall be bound, before taking out such a license, to deposit certified copies of their laws, with the exception of companies to be established under the laws already enforced, and shall further be bound to choose *domicilium citandi et executandi* in this State, and to give notice thereof in the *Staatscourant*.

7. Each assurance company as aforesaid shall, at the end of each financial year of such company, send in to the Treasurer-General a certified statement, together with all other statements which may be required by this law, giving the total amount of assurances in force, the total amount of new assurances, commencing with the premiums

received thereon during the year, and policies which have lapsed during the year, and shall further be bound to send to the Treasurer-General a certified statement of the profits and commissions of such company, together with copies of the balance sheet and income statement for the year, all having reference only to the business of the company transacted within this Republic.

8. Each company which transacts assurance business within this State shall within three months after date of this law or beginning of their business or alteration of the official, as the case may be, publish in the *Staatscourant* the name of the head official, manager, secretary or agent in this State, and the place where the head office of the company may be situated as required where at the time the *domicilium citandi et executandi* of such company shall be established.

9. Each such head official, manager, secretary or agent, whose appointment has been published in the *Staatscourant*, as aforesaid, can, in the name of the company which he represents, sue or be sued on all notices, summonses, or other legal proceedings, which shall be accepted by such company if the same be served on such head official, manager, secretary or agent, or be left in the hands of any servant of the company in the head office, or if, in the absence of such person, the same be attached to the front door of such head office.

10. If any documents under the provision of this law required to be produced shall be false in important particulars, the person who has signed or delivered the same shall be liable to a fine not exceeding £100, or imprisonment with or without hard labour for a term not exceeding six months.

11. Any company which omits to send in any statements required by this law for a term of three months or more may, by request of the State Attorney and after notice by the High Court, be prohibited from transacting business in this State, whether for an indefinite period or for such time as the court may determine.

12. It shall be the duty of the Treasurer-General to see that each company coming under the provisions of this law, properly fulfil the provisions thereof; and he shall be bound to submit to the Government yearly, a report of the particulars of the business of each such company according to the statements required by the law.

13. All charges outstanding against policies issued in this State shall be paid, if such payment is desired, by the person entitled to the assurance money in this State, and the receipt of the executor or other person administering the estate shall sufficiently prove that such company has duly paid the amount of the policy.

14. The security under Article 1 of this law shall be placed as a security to the Government and the inhabitants of this State for the good faith and honourable business transaction of each such company, and such security shall be held in case of any judgment by any qualified court against such company.

15. No such company shall, in any policy yet to be issued in this State, make any conditions which conflict with the provisions of military or commando law, and shall in no case approve of such condition or plead against the payment of such policy if the person assured dies in military or commando service.

16. If any agent, director, secretary, manager or representative of one of the herein-named assurance companies infringes or fails to carry out one or more of the provisions of this law, he shall, as such official, be liable to a fine of not more than one thousand pounds sterling, or imprisonment with or without hard labour for a term not exceeding six months, together or separate.

17. The Courts of Landdrost and Mining Commissioner shall have jurisdiction for all infringement except that of Article 11.

18. This law shall come into force on the 1st of January, 1893.

S. J. P. KRUGER,
State President.

DR. W. J. LEYDS,
State Secretary.

Office of the Government,
Pretoria, 30 *July*, 1892.

RECEIPT FORM A.

Received from the
by Mr. the sum of
pounds sterling, being security required by Article 1 of Law
1892, which amount the Government of the South African Republic hereby declares to have received in trust from the
and promises to pay back to the said company at this office within three months after they have given notice in writing to the Government of the South African Republic that they wish their business in this State to end, and have clearly proved to the satisfaction of the Government that they have now no obligations to anyone living in this State.

The interest on the amount named shall be 3 per-cent per annum, to be paid yearly by the Treasurer-General at that office to the representative of the assurance company before or on the 1st of February of each year.

The assurance company have the right, on giving three months notice, to change these securities in cash into sufficient securities approved by the Government, for which a receipt will be given according to Form B, as required by the regulations contained in Article 2 of this law.

Pretoria 18

RECEIPT FORM B.

Received from the
by Mr. the following documents, to wit

doing for security mentioned in Article 1, of Law 1892, and the Government of the South African Republic hereby agrees to keep them in safety for the said assurance company, and to return them thereto, either in exchange for other documents of equal value, by approval of the Government, or for cash, in case the assurance company so desires, or after the said company has given written notice to the Government of the Republic, that they wish their business in this State to end, and have clearly proved to the satisfaction of the Government that they have now no obligations to anyone living in the State.

The representative of the assurance company is entitled to receive the interest on the above-mentioned securities.

Pretoria 18

REVIEW.

*Distribution of Diseases in Africa.**

The subject of extra premiums to cover the risks of foreign residence, though of great interest and of growing importance in these days of almost universal travel, is at present in a very early stage of development. The practice of various offices differs considerably in the treatment of these risks, and in many cases there are little or no data available to form the basis of a scientific estimate of the risk actually incurred, so that a premium has often to be fixed more by guesswork than by any scientific process. Hence any contribution to our knowledge of the effects of climate on the constitution and prospects of life, should be welcomed by the actuarial world; and for this reason, the book recently published by Dr. R. W. Felkin deserves mention in the pages of this *Journal*. The subject is too large and complex to be satisfactorily dealt with as a whole, and the best, if not the only way, of approaching it, is to take each separate climate by itself, and to obtain all possible data regarding it; then, having considered these carefully, an attempt may be made to form a sufficiently accurate estimate of the climate risk for practical purposes. In the work in question, the author has confined himself to Africa, as is evident from the title, *On the Geographical*

* *On the Geographical Distribution of Tropical Diseases in Africa:* by R. W. Felkin, M.D., F.R.S.E., F.R.G.S. 1895.

Distribution of Tropical Diseases in Africa; but he has made it abundantly clear that there are so many different climates there, conducive in such widely varying degrees to different tropical diseases, that there is very great scope for investigation as to the different rates of mortality in this one continent alone. He has dealt more particularly with the medical aspect of the question, and the work is therefore more likely to be of use to the medical officers of insurance companies than to their actuarial advisers; but there are many points of interest to the latter contained in the book, and I have endeavoured to bring these out in the following summary.

The author points out that the subject of acclimatization is one that is very largely misunderstood, and exposes the fallacy of the popular idea that a European migrating to a foreign country becomes "acclimatized" in the course of a few years. He explains that the supremacy of white races in the tropics is only maintained by a constant stream of fresh immigrants, without which it seems that the race degenerates to a considerable extent, owing to the effects of the climate. His view is that the process is one requiring whole generations, not merely a few years; and that Europeans can only become thoroughly acclimatized in Africa if migration takes place step by step spread over a very long series of years, and if great care is exercised in the selection of persons going there. It appears that climate has a great effect in changing the physical stamina and mental characteristics of emigrants; and that in Africa in particular the enervating climate causes immigrants from the north to become lazy, indolent, and to some extent emasculated, undoubtedly changing the national character.

The work should prove particularly useful to medical advisers of insurance companies doing business in tropical countries, as Dr. Felkin indicates the types of lives which are best suited to resist the adverse influences of the climate, and the special diseases prevalent in the tropics. For example, persons with the bilious temperament are, he states, best suited to withstand the climate, as they possess great physical endurance, and are not so liable to be prejudicially influenced as persons with the sanguine temperament. The lymphatic temperament is the least suited to the tropics, and he emphatically recommends all persons of this type to stop at home, as also all with a syphilitic, rheumatic, scorbutic, or malarious history, or with a tendency to gout or heart disease. Again, no immigrant should be younger than 25, as persons under that age suffer more from typhoid, dysentery, and the severer forms of malaria. Finally, south Europeans are better suited to and can stand the climate better, than those from the north of Europe. Thus the French are rapidly colonizing Algeria, the Arabs are finding their way to the Soudan, and the Soudanese into Bantu Africa, showing a steady stream of migration to the southward; and Dr. Felkin believes that in time Europeans may be able to colonize even Central Africa and become acclimatized there. He points out that to a large extent even the worst climate in Africa, may be made less unhealthy by a strict attention to sanitary measures; and as an example of this, he states that a careful supervision of the water supply of Sierra Leone has had a marked effect in lowering the death-rate. Sex has, he thinks,

no special influence on the power of withstanding the climate, but no woman with a tendency to the diseases special to her sex should emigrate to tropical Africa. Children apparently do well there for three or four years, but must then be sent to a temperate climate, or they would degenerate physically and morally; and in this respect there is an evident analogy to the well-known characteristics of the Indian climate. It should also be noted that persons going to Africa for the first time should arrange for their arrival to take place at the coolest season of the year, and that this varies considerably in different parts of the Continent.

It is clearly shown that altitude must be an important factor in considering the proper extra premium to be charged for any locality, for the continent may be divided into three zones of climate: the first or hot zone, from the sea level up to about 3,000 feet, which is the region of tropical diseases; then the temperate zone, from 3,000 up to 12,000 feet, in which the climate is less pernicious; and lastly the cold zone, above 12,000 feet, which is incomparably the most healthy.

After considering the general effect of the African climate upon the health and system of European immigrants, the author passes on to review in detail the characteristics of eight different regions into which he has divided the Continent, each of which has approximately the same climatology throughout itself but differs from all the others. These are discussed at length from a medical standpoint and may be summarized as follows:

I. *North Africa.*—Here a gradual acclimatization of the French appears to be taking place; and, doubtless, Italians and Spaniards would fare equally well. There is a considerable amount of phthisis present, especially in the towns, but this may be due to the number of patients sent there in the hope of a cure. Incidentally, it may be noted that it is a mistake to send patients advanced in that disease to Algeria, though the climate is very suitable for those in whom it is only incipient or in an early stage.

II. *Nile Valley.*—The health of this district depends entirely on the river, which is so habitually polluted that the people practically imbibe a solution of filth during the summer months when the water is low. This accounts for the infant mortality, which is so heavy that nearly 50 per-cent of the children born die before attaining the age of five years—a rate more than two-and-a-half times as great as that prevailing in this country. It would be possible to effect a great improvement in this respect and purify the water supply, as there are no insuperable engineering difficulties in the way; but the cost would be enormous.

III. *East Africa.*—The East Coast is undoubtedly very unhealthy, and, with a few exceptions in the interior, immigrants from temperate climates cannot thrive in this district. There are only one or two areas, such as the Shiré highlands, where white races can colonize. The mortality among Europeans appears to be due in large measure to malaria, which is contracted by them during the stay on the coast which is necessitated by the difficulties of transport to the

interior. As malaria is exceedingly rife on the coast the system becomes thoroughly impregnated with the poison, and for this reason the author advocates the construction of light railways to carry travellers rapidly across the malarious belt. If this were done, he considers that the mortality might be expected to diminish to a large extent.

IV. *West Coast.*—This is the hottest and wettest region in the whole globe, and as is well known, the climate is deadly to Europeans, though much can be done to ameliorate it by proper sanitary measures, as in the case of Sierra Leone already mentioned. The Portuguese part of the coast is not so unhealthy as that farther north; but the only place in the district that is at all healthy, is the Cameroon mountain system. It is interesting to note that at one season of the year a dry wind, called the Harmattan, is prevalent; and during this the decay of vegetable matter is arrested, and in consequence, the spread of malaria is checked. So marked is the effect of this wind, that patients suffering from malaria rapidly become convalescent while it is blowing, and it has a very beneficial effect on small-pox. Curiously enough, it seems that persons vaccinated during the Harmattan are not protected against the latter disease, as the vaccination will not take.

It appears from the author's statements, that the enormously high death-rate in this district is not due to malaria alone, but is largely due to the fact of diseased persons proceeding to Africa, also to a want of knowledge of the proper precautions to be taken to preserve health, and a wilful neglect to lead a well-regulated life. A few figures are mentioned here showing that the average death-rate in Gambia and Lagos for a period of 10 years ending in 1888, was approximately 10 per-cent; and it is stated that the mortality in Sierra Leone was about 42 per-cent. Unfortunately it is not quite clear what is meant by "the average death-rate", but apparently the author means that the deaths in 10 years were 42 per-cent of the average population, which, of course, gives an annual mortality of about 4 per-cent. This is consistent with the figures given for the Gold Coast, which bring out an annual mortality among the Government officials of about 5 per-cent during the same period. One important point should be noticed, namely, that these officials appear to have better prospects of life than the rest of the white population on this coast—the mortality among the latter being at the rate of about 8 per-cent per annum. This is probably due to their obtaining more frequent and longer periods of leave, it being the custom, I understand, for the officials to have at least two months' leave each year, which can be extended to three or four months on a very slight excuse.

V. *Sahara* and the *Soudan* are dismissed in a few words, owing to the absence of statistics. The author, however, states that malaria is almost absent, except at the oases, and that the district is, on the whole, healthy.

VI. *Equatorial Central Africa.*—This is one of the most unhealthy parts of the Continent, owing to the mean annual range

of temperature being very small (less than 5° Fahrenheit in the Congo region); to the great annual rainfall (about 50 inches), and great humidity; and the prevalence of malaria throughout the entire district. Enteric fever also occurs, with a frequency varying with the habits and customs of the natives and their sanitary surroundings, rather than with the character of the country; and it is an interesting fact that, generally speaking, this disease does not affect those localities where the banana is the staple food of the population. It is far more frequently found where the people live chiefly upon grain.

It is important to notice that the Congo region is extremely unhealthy, in view of the fact that it is being colonized to a considerable extent by the Belgians; and, as the white population increases, it will become more necessary for insurance companies to turn their attention to the risks of that climate. The Central Congo district has such a bad name that the mortality among Europeans resident there is said to be about 25 per-cent, and though there does not appear to be much foundation for this report in the shape of available statistics, yet it may be taken as an indication of an unusually heavy death-rate.

VII. *South Africa.*—It is scarcely necessary to notice this, beyond stating that it is generally healthy, and has a climate resembling that of the temperate zone. A large portion of the district is included within the free limits of the leading insurance companies; and as the tendency of the present day is rather to relax these limits than to increase their stringency, the question of the correct theoretical extra premium to be charged for this region need not be investigated.

VIII. The last of the divisions comprises *Madagascar*, the *Seychelles*, and *Mauritius;* and it is scarcely necessary to say more of these than that malaria is prevalent in the first named, especially on the coast, and also in Mauritius, although, curiously enough, that island was quite free from the disease up till 1866.

The remainder of the book is taken up with matters relating exclusively to the medical profession, being a description of the nature of various tropical diseases, and the treatment required for each. This is not strictly within the province of the *Journal*, and would require a special knowledge of the subject to review it adequately.

An appendix to the work contains a map and a table constructed by the author on a novel and ingenious principle. His plan is to denote the various diseases by symbols, and the prevalence of any disease or the mortality therefrom, is shown by the number of times the symbol is repeated. Thus, an isolated symbol in an area denotes that the corresponding disease is present; if the symbols occur in pairs it is very prevalent, and if in groups of three the mortality from it is excessive. Malaria (which appears, in Dr. Felkin's view, to be the chief cause of the heavy mortality, as it is treated more fully than any

other disease) is indicated by a pink shading, and its intensity or degree of prevalence is naturally denoted by the depth of colouring, the darkest shades being used for the districts where the mortality from malaria is greatest. Hence when any district contains numerous symbols occurring in groups of three, and when the colour of that part of the map is a deep red, it is seen that persons resident in that area are subject to the risk of many diseases including malaria, and that the mortality from these is excessive. On the other hand, the healthier parts of the country are at once seen by the paper being lighter in colour and by the absence of symbols or their occurrence singly instead of in groups. From this map, therefore, a comparative estimate of the healthiness of the various regions can be obtained at a glance, and a rough idea gained of what would be a suitable extra premium for the corresponding climate risk; and it should be of assistance to actuaries when considering the rates to be charged for residence in Africa. The table is constructed on a somewhat similar principle, and will be useful if it is desired to find out at a glance in what parts of the Continent any particular disease is most prevalent.

The general conclusions regarding the work are that, as already mentioned, it is of greater value to the medical officer than to the actuary; but owing to the map and the graphic notation used thereon it should prove useful to anyone desiring information as to the distribution of disease in Africa and the relative mortality in the different parts of the Continent.

A. E. SPRAGUE.

ERRATA AND ADDENDA.

*System and Tables of Life Insurance.**

Revision of Life Insurance Formulas, Return Premiums and Survivorship Annuities.

By Levi W. Meech.

Mr. LEVI W. MEECH has issued a memorandum setting out certain revisions and alterations which he desires to make in his volume, *The System and Tables of the Thirty American Offices;* and in view of the many references that have lately been

* A treatise developed from the experience and records of Thirty American Life Offices under the direction of a Committee of Actuaries, by Levi W. Meech, Actuary in Charge, 1881, and Revised Edition, 1886.

made to this work, we believe our readers will be glad to have these emendations available for reference. They are accordingly reprinted in the exact terms of Mr. Meech's note.

ED. *J.I.A.*

In arranging a table of rates on the *Return Premium* plan, the computer may observe a difference in the formulas of the text-books. In most of the formulas, the loading denoted by g enters into the denominator, thus requiring the loading of the office premiums to be fixed in advance, and unfitting the results for future adjustment. The earlier work of David Jones deals only with the net premiums (pages 206, 207), leaving the office loading to be applied afterwards. In departing from this method, the latter solutions involving g appear at fault, although skilful changes of numeric results may diminish the error. After critical examination of the two species, it is believed all actuaries will admit that a return to the former method is indicated. Accordingly, the following revision or change is to be inserted in the *System and Tables of the Thirty Offices:*

Page 253.—On the lines 15-26: In "net annual" (line 15), change "net" to gross. Prefix the factor g to P, P_x and M_x, which occur in ten places (except within parentheses). Cancel g from the three denominators. On lines 4 and 6, change $x+m$ to $x+m+1$.

For example, a person aged forty years buys a life annuity to be deferred fifteen years, and to be paid for by fifteen annual premiums. By the Thirty Offices' Table and 5 per-cent interest, the correct net annual premium for each \$1 per annum will be \$0·4081. With the condition that all premiums paid shall be returned without interest, in the event of death during the fifteen years' deferment, the correct net annual premium would be \$0·4479 on each \$1 of annuity.

Page 230.—Errata—Year 95 and $3\frac{1}{2}$ per-cent. Change 0668 to 6668.

Page 260.—In the table for Equal Ages 64, the 8 11.7 should be 7 11.7.

Page 272, line 13.—Change P_{44} to $P_{44\cdot 50}$; change 12·599 to 11·486; also change 83689 to 91799, and 815·55 to 410·05.

Page 295.—Erase or substitute for lines 7-13 as follows:

Survivorship Annuities.—Here the values of b' and of b'', which were constant for two lives, were found by F. E. Colenso (*J.I.A.*, xxxi, 337) to be variable for three lives. Still their values might be generalized from numeric examples (solved by the exact series on

page 283, line 8), were a sufficient number available. But the admirable system of balancing annuities, proposed by Prof. De Morgan, which so readily gave the formula of the Thirty Offices' System and Tables, page 283, line 7, from problems 3 and 4, has led to the following improvement: Among x, y, z, let g denote the greatest and l the least, age. The required annuity may be denoted by the Institute notation $a^{\,1}_{yz|x}$ or by ${}_{yz}'a_x$:

$$ {}_{yz}'a_x = (a_{xy} - a_{xyz}) + m[a_x - a_{xz} - (a_{xy} - a_{xyz})]. $$

When z exceeds y, $m = \frac{1}{2} + \dfrac{0{\cdot}1\frac{1}{3}(g-x)}{g-l}$. And if y exceeds z, $m = \frac{1}{2} - \dfrac{0{\cdot}1\frac{1}{3}(g-x)}{g-l}$.

$$ {}_{zy}'a_x = a_x - a_{xyz} - {}_{yz}'a_x . $$

The last equation is found by interchanging y and z in the exact series on page 283, line 8, and taking the sum of results compared with page 280, line 5. Proceeding now to Examples: First, let the age of the annuitant x be 45 years, of the decedent z 60, and of the survivor y 30. Secondly, let x be 30, let z be 60 and y 45. Thirdly, let x be 47, let z be 30 and y 23 years. By the Thirty Offices' Table, with interest at 4 per-cent, the present value of the required annuities found by the exact series is 5·1742, 6·9146 and 1·2815 in order. The corresponding values found by the formula above are 5·1755, 6·9100 and 1·2843.

The last solution becomes important in view of Mr. Makeham's statement in the *Journal*, vol. x, p. 241: "If we should ever possess a complete table of the values of the simple survivorship assurance A^{1}_{xyz}, and also of the annuity ${}_{zy}'a_x$ we should be able to calculate the numeric value of double survivorships in any case with the greatest facility."

As Mr. Makeham observed, the study of the "Survivorship Problems" is adapted to a new field of life insurance and annuities, including no small share of wills and probate business.

Final Series.—Pages 35–38 and 12*a*–12*c*.* By changing s to v' in Makeham's law, the common l takes the form of commutation D. Thus the D columns of select Tables for every initial age are virtually combined into one Life Table by the method of Final Series. Its superior accuracy is shown by the Thirty Offices' Table applied to all the discussions of office experience published in this country during the last ten years.

* These pages appear only in the Revised Edition of 1886; the pages not so marked refer to both Editions.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Distribution of the Divisible Surplus of a Life Assurance Company, with special reference to the Method originated by Dr. Sprague and other Methods derived therefrom. By George J. Lidstone, *Fellow of the Institute of Actuaries, Assistant Actuary of the Alliance Assurance Company.*

[Read before the Institute, 25 March 1895.]

Introductory Remarks.

1. Some time ago, the writer had occasion to examine the principal methods adopted by life assurance companies in the distribution of surplus, and particularly the plan which is known as "Sprague's Method", from the name of its inventor, Dr. Sprague, or the "Equity and Law Method", from the name of the company to which it was first applied. The plan is at present employed by comparatively few British offices, but it has recently been adopted by several companies in substitution for systems previously in vogue, and it appears to be rapidly growing in theoretical importance as a standard by which to test the effect of other systems of distribution. Nevertheless, it would appear that, apart from the brief description of the method originally given by Dr. Sprague (and quoted below), there is no systematic discussion of the principles of the method in the pages of the *Journal of the Institute of Actuaries*, and it is therefore thought that a few notes respecting some of the points

which arise in connection with the practical application of the method may be of service to those who are studying the subject, but who may share the experience that little information is to be gleaned from the *Journal*.

2. In the following remarks, it is not intended to give any strong expressions of opinion as to the exact system which should be followed in any special case, but rather to indicate and discuss some of the questions which are likely to arise for consideration. In actuarial work, it is almost impossible to lay down unbending rules which can be universally employed without modification, and this is especially the case with the subject of distribution of surplus, in reference to which our theoretical conclusions must often be modified by a variety of other considerations, arising out of the special circumstances of the office in question—such, for example, as the effect of methods which may have been followed in the past, and the representations which may have been made with regard to the future.

DR. SPRAGUE'S DESCRIPTION OF HIS PLAN.

3. It will be desirable to quote at once, *in extenso*, Dr. Sprague's description of his plan. He says, in an editorial note (*J.I.A.*, xiv, 396): "This appears a convenient opportunity "for stating briefly the method of distribution of profits which "appears to ourselves to combine in the highest degree the "requisites of justice and facility of application, and which we "have found in practice to give very satisfactory results.

"If the average rate of interest at which the total funds, "inclusive of bankers' and agents' balances, and other unpro-"ductive assets, are improved, exceeds the rate at which the "valuations are made, it is clear that a profit will be realized in "the nature of excess of interest on the amount of the funds at "the last valuation, to which profit the new members have "contributed nothing. Let the amount of profit so earned by "each of the old assured still remaining on the books be "ascertained and appropriated to his policy. For example, if "the valuations are made at 3 per-cent interest, and the average "rate realized has been £4. 8*s*. per-cent, then the reserve made "for each participating policy at the last valuation (five years "ago) is to be multiplied by $\cdot0809=(1\cdot044)^5-(1\cdot03)^5$; and the "product will be the profit to be in the first instance appropriated "to the policy. The sum of all these amounts being found and "subtracted from the surplus divisible among the assured, there

"will remain a sum which may be fairly divided among all the "assured in proportion to the premiums (without interest) they have "respectively paid since the last valuation. As regards persons "of the same age at entry, it is clear that, apart from the effect of "selection, the profit on their current premiums must be nearly "the same whenever their policies were effected; and if the "premiums are loaded with a percentage on the net premium, or "approximately so, the distribution of the surplus in proportion "to the premiums paid will give very fair results. This method "will have the effect of giving larger cash bonuses to the policies "the longer they have been in force, but not unreasonably or "unfairly so."

4. It will be seen that the fundamental principle underlying the method is to make a broad division of the surplus into two parts, which are distributed according to different systems. The first part, consisting of interest earned in respect of the reserves under participating policies in excess of the rate assumed in the valuation, is divided in proportion to the reserves upon which that excess interest has been earned. The second part embraces the whole of the remaining surplus (including the profit from loading, lapses, surrenders and other miscellaneous sources of surplus, increased or diminished by the profit or loss in respect of mortality), and is divided in proportion either to the tabular premiums paid, or (in some modifications of Sprague's plan) to the loadings paid. In its broad outlines, the method may thus be said to follow Mr. Sheppard Homans' "contribution method", which is based upon an elaborate analysis of the sources of profit (see *J.I.A.*, xi, 121); but, while preserving to a great extent the advantages of that method, Dr. Sprague's plan avoids the complicated and somewhat cumbrous processes in consequence of which, and of certain theoretical objections, Mr. Homans' method has failed to receive much support from British actuaries.

Modification of Dr. Sprague's Plan—The Loading System.

5. Several companies which have adopted the fundamental principles of Dr. Sprague's method of distribution have introduced changes of detail. Of these changes, perhaps the most important is that of substituting the loading for the office premiums as the measure for allotting the second portion of the surplus. It will be found that the practice of offices is equally divided on this point, for out of eight British offices which have

adopted Dr. Sprague's principles, four adopt the loading system and four the premium system. If the loading be a constant percentage of the net premiums at all ages, the two methods will obviously give identical results, but this will not usually be the case in practice. Dr. Sprague himself, in his description of his method, says, "*If the premiums are loaded with a " percentage on the net premiums, or approximately so,* the " distribution of the surplus in proportion to the premiums paid " will give very fair results", from which it seems fair to infer that if this condition were not fulfilled he would consider some modification necessary or desirable. This view is confirmed by some subsequent remarks by Dr. Sprague, appearing in his opinion on the case submitted to him by the Australian Mutual Provident Society. Mr. Black, the actuary of the society, had suggested the substitution of loading for premiums, and, commenting on this, Dr. Sprague said: "Considering the " manner in which the premiums of the society are adjusted . . . " and the much heavier proportionate loading at the younger " ages, there seems little doubt that the method proposed by " Mr. Black is better suited for adoption by the Society than " that to which my name has been attached."

Discussion of the Loading System.

6. Proceeding to consider the rationale of the loading method, we shall find it convenient to specify in detail the principal sources of surplus, which can be most conveniently exhibited as follows in the form of a

Profit and Loss Account.

1. Profit derived, in respect of participating policies, from interest on the reserves earned in excess of the rate assumed in the valuation.	1. Amount absorbed (if the valuations be made according to the H^M and $H^{M(5)}$ Tables) by passing from the H^M Reserves to the H^M and $H^{M(5)}$ Reserves in respect of business in its second quinquennium of policy existence.
2. Loading received in respect of participating policies.	2. Expenses of management and commission.
3. Profit from surrenders and lapses.	3. Miscellaneous losses.
4. The whole of the profit from the working of the non-participating business.	4. Loss from unfavourable mortality (if any).
5. Miscellaneous profits.	5. Surplus.
6. Profit from favourable mortality.	

7. The first two items of profit—namely, surplus interest and loading—can, without difficulty, be allotted to the several policies in the proportions in which they have been contributed.

8. The total of the items of profit classified under headings 3, 4 and 5, less miscellaneous losses under heading 3, represents what may be called "trading profit", which arises from the general working of the business, and can hardly be said to belong in any definite proportion to the continuing participating policy-holders. On the other side of the account we have "expenses of management and commission", an item which it is also extremely difficult to apportion with anything approaching strict equity. Under these circumstances, it appears to be a suitable and convenient course to set off the miscellaneous profits under headings 3 to 5 against the cost of working the business and the miscellaneous losses—headings 2 and 3. The balance of profit or loss, as the case may be, will probably be relatively small (*see* paragraphs 15, 20 and 21), and may well be divided in proportion to the loading contributed, since that is primarily set aside as provision for expenses and contingencies, and therefore appears to be a fair measure for the allotment of the balance of profit or loss referred to.

9. The profit derived from favourable mortality will probably be due in a considerable measure to recency of selection (or as it is sometimes called "Suspended Mortality"), and if the valuation be made by the combined tables, we may reasonably apply such profit in the first instance to provide for the amount absorbed, in respect of policies more than five but less than ten years in force, by passing from the H^M reserves to the H^M and $H^{M(5)}$ reserves.* The remainder of the profit from mortality (or the whole thereof if the valuation be made according to a single table) could, by a somewhat elaborate process, be divided among the participating policies, arranged in groups according to the attained age of the lives assured, in proportion to the difference between the expected

* It may be of interest to remark, in passing, that the average amount of the additional reserve thus made, in the case of an office valuing at 3 per-cent, will be about 1 per-cent on the sum assured. This may be shown by the use of Mr. King's hypothetical office. Thus, in the case of an office 10 years old, we find from Mr. King's tables:

Reserves H^M and $H^{M(5)}$ 3 per-cent .	82,951	Number of policies in force in second quinquennium of existence as per table X (*J.I.A.*, xx, 274)
„ .	78,223	
Difference	4,728 ÷	429,543
		= 1·10 % nearly.

death strain and the actual death strain experienced during the valuation period in respect of each group; but it would appear to be necessary, in order to avoid very undesirable irregularities in the resulting allotment of surplus, to introduce some graduation of the experienced rate of mortality before applying such a process. A method of this kind may be considered necessary when the mortality actually experienced differs very widely from that assumed in the valuation, as, for example, in the case of the Mutual of New York, which, at the time Mr. Sheppard Homans first introduced his "contribution" system of distribution, had experienced only 70 per-cent of the assumed rate of mortality. It may, however, be doubted whether so much refinement is necessary, or even desirable, in the more usual case when the table used in the valuation gives a fair representation of the rate of mortality likely to rule in the future, with perhaps a slight margin for contingencies. In such a case it would seem to be sufficient to divide the balance of mortality profit in proportion to the amounts of "death strain at risk", namely:

[Sums assured and bonuses] *minus* [Corresponding reserves].

The effect of this method will evidently be to give a smaller share of the surplus to policies of long duration than to policies recently effected, and the same result would be produced by the more elaborate plan referred to above unless the function

[Expected rate of mortality] *minus* [Actual rate of mortality]

increases more rapidly with the age than the quantity

[Death strain] ÷ [Amount at risk]

decreases. This result, while it may not be objectionable from a theoretical standpoint, will probably be considered undesirable in practice, especially in view of the fact that the expenses and commission press most heavily on the new business. An adjustment which, although rough, will be in the right direction may be obtained by dividing the balance of the profit from mortality, after making the initial deduction referred to in paragraph 8, according to some system which does not give a smaller share to policies of long duration than to policies of short duration. The desired result will be obtained if the balance of profit referred to be divided in proportion to the sums

assured, the office premiums or the loading. It will not greatly matter which of these methods is selected, but it would be proper that the same plan which is adopted for the division of the mortality *profit* shall be used to apportion any *loss* which may arise from unfavourable mortality in any valuation period. If the last method be adopted we shall allot the balance of mortality profit in the same proportion as the miscellaneous profits and losses, and we shall find that we have arrived at what has been called the "loading method", by reasoning which, although it may be said to be to some extent *a posteriori*, does not involve any great departure from principle.

Basis of Calculation of Loading.

10. In connection with the loading method, there arises for consideration a very important question which goes to the root of the whole matter, namely, on what basis should we calculate the net premium which is used in determining the loading? The most obvious course, and that which appears to be commonly adopted in practice, is to employ the net premium which is used in the valuation. As far as the rate of interest is concerned, it certainly seems proper to adopt that used in the valuation, since the excess interest is returned in the same proportion which it has been contributed, but it is not so clear that in other respects the valuation basis should be rigorously followed.

11. Take, for example, the common case of a company valuing its business by the combined H^M and $H^{M(5)}$ Tables. While this method practically gives very good results it is difficult to see that it has any scientific rationale, and it must be regarded as simply a convenient machinery for arriving at reserves which are very close to, but in most cases somewhat more stringent than, those which would be produced by the use of analyzed tables, such as Dr. Sprague's and Mr. King's. As Mr. King very forcibly and clearly puts it (*J.I.A.*, xxxi, 251), "For valuation " purposes an aggregate table was of use only in so far as it gave a " good approximation to the reserves that would be made by the " select tables. It was only because he found by practical " examination that the select tables prepared by Dr. Sprague did " very closely approximate to the result of the H^M and $H^{M(5)}$ Tables " that he used the latter combination. If they had differed very " much he should unhesitatingly have thrown aside the combined

"tables and adopted the select." It is now generally recognized that the select tables give a more accurate measure of the risk premiums than is given by an aggregate table. It is therefore submitted that, in the division of surplus according to the method now under discussion, we shall obtain a more equitable apportionment as between the entrants at various ages by the use of the "select loading" than by the use of the "H^M loading", and if this be admitted, it seems evident that it is not right to disturb the whole incidence of the distribution merely because the valuation has not been made rigorously on the basis of the select tables, but by a method which gives a convenient approximation to the select reserves.

12. The same argument applies with slight modifications in the case of a company making its valuations on the basis of the H^M Table. The adoption of a basis giving weaker reserves than would be made by the use of the combined tables can have but very little influence on the real profit-earning power of the company, its principal effect being to throw a slightly larger amount into surplus during the early years of assurance and a correspondingly smaller amount during the middle and later years of assurance. In the case of a well-established company doing a normal amount of new business it may easily be seen that there is only a small difference between the amount of surplus periodically disclosed by a valuation according to the H^M Table and the corresponding amount which would be disclosed if the valuations were made throughout by the combined tables. For example, if the office has reached a stationary condition, the new business being just sufficient to replace the "waste", the difference between the reserves will be constant and the annual interest on the difference will represent the amount by which the profit annually disclosed will differ from the corresponding amount in an office, similar in all other respects, but valuing by the H^M and $H^{M(5)}$ Tables. This will in any case bear but a small proportion to the total surplus, and it will be reduced in the event of the accession of a larger amount of new business than is required to maintain the office in a stationary position. It seems a fair conclusion that as the use of a weaker valuation basis makes comparatively small difference in the surplus it is not a sufficient reason for adopting the H^M loading in lieu of the select loading, a course which would involve a very considerable alteration in the relative amounts of surplus allotted to young and old entrants respectively.

13. It may be urged that if the select loading be adopted in the distribution we should, for the sake of consistency, use the select reserves as the basis of the allotment of the surplus interest, and that if this be not done our formulas for the cash surplus allotted to the various policies will be made up of heterogeneous elements. From a theoretical point of view it must be admitted that such an argument would be well founded, and if extended tables of policy-values, calculated strictly according to the analyzed tables, were available it would certainly be desirable to employ them. As a matter of fact, however, such tables are not available, and we have to be content with approximate accuracy, and, under such circumstances, a much closer approximation will be obtained by the use of the hybrid formulas than by the use of the more symmetrical, but less correct, formulas involving the H^M loading.

Summary of the Results of the Loading System.

14. The operations of the loading method may be briefly summarized as follows:

(*a*) The profit derived from surplus interest is divided in proportion to the reserves in respect of which the interest has been earned.

(*b*) The whole of the loading paid in respect of each policy is next returned.

(*c*) The remainder of the profit derived from all sources is set against the expenses of management and commission and the losses of all kinds, the balance (profit or loss, as the case may be) being applied as a *pro rata* increase or decrease in the amount allotted under heading (*b*).

15. Now it will be seen that the premium method and the loading method practically agree in adopting the process of setting off the miscellaneous profits against the miscellaneous losses and the expenses of management, and it will be shown below that the difference (which is divided in the one case in proportion to the premiums paid, and in the other case in proportion to the loading) is, in the case of an average first-class office, comparatively small. There will thus be but little difference between the results of the premium method and the loading method in the case of the amount allotted under heading (*c*), the

principal distinction between the two methods being that the latter returns all the loading in the proportions in which it has been contributed, while the former *throws the whole of the loading into one fund, which is then re-distributed in proportion to the premiums paid.* This course seems to be clearly inequitable (except in the unusual case of the premiums being loaded with a constant percentage, when the results of the two systems are coincident), and it would therefore appear that, to use the words of Dr. Sprague, the loading system is "better suited for adoption" than the premium system.

16. It may be desirable to consider somewhat more closely the reasoning of paragraph 8—to the effect that the miscellaneous profits on the one hand and the expenses of management and commission on the other, being very difficult to apportion between policyholders of different classes, a simple and not unfair way out of the difficulty is to set off one against the other. It will probably be admitted that the process in question is legitimate, so far as the general expenditure—excluding initial commission and expenditure, and all renewal commission—is concerned, but it may be said that the latter items are capable of being measured with approximate accuracy, and that the method of distribution should take account of this. In order to investigate this point, let us assume that the commission is at the common rate of £1 per-cent on the sum assured in respect of the first year's premium and $2\frac{1}{2}$ per-cent on renewal premiums, and that the initial expenditure, exclusive of commission, amounts to 1 per-cent on the sum assured. Spreading these amounts over the whole duration of the several policies, they will be equivalent to a uniform annual payment of

$$\frac{2+2\cdot 5\pi a}{\text{a}} = \frac{2-2\cdot 5\pi}{\text{a}} + 2\cdot 5\pi = \text{E, say}$$

where π is the gross premium. The values of this expression will give for different ages at entry what has been called the "equalized pressure" of that portion of the expenditure with which we are now dealing, and they will form a fair basis for the apportionment of such expenditure.

17. In order to obtain numerical values of E, it will be necessary to fix upon a representative scale of office premiums, which will also be required in subsequent investigations. The following table exhibits for quinquennial ages rates which it is thought may be taken as fair average ones.

Table A.—*Showing the assumed Annual Rates of Premium for an Assurance of £100, with Profits.*

Age at Entry	Whole-Life Assurance π_x	Endowment Assurance, payable at 60 or at previous Death $\pi_{x:\overline{60-x}}$	Age at Entry
20	1·950	2·450	20
25	2·175	2·850	25
30	2·450	3·400	30
35	2·800	4·150	35
40	3·225	5·300	40
45	3·800	7·200	45
50	4·550	11·050	50
55	5·525	...	55
60	6·900	...	60

18. Using the values of a according to the select tables at 3 per-cent we shall obtain, from the rates above given, the following table, showing the values of E for different ages at entry and also the values of the ratio

$$E \div \phi$$

where ϕ represents the loading on the same basis.

Table B.—*Showing the Annual Amount* (E) *equivalent to an initial expenditure of £1 per-cent with a commission of £1 per-cent on Sum Assured and $2\frac{1}{2}$ per-cent on Renewal Premiums; and the Ratio of that amount to the Total Loading* (ϕ).—(*Select* 3 *per-cent*).

Age at Entry	Whole-Life Assurances		Endowment Assurances at 60		Age at Entry
	E	$E \div \phi$	E	$E \div \phi$	
20	·136	·352	·156	·344	20
25	·144	·305	·171	·300	25
30	·155	·295	·193	·287	30
35	·169	·290	·223	·286	35
40	·187	·300	·268	·284	40
45	·210	·303	·342	·286	45
50	·240	·302	·486	·270	50
55	·278	·313	...	...	55
60	·332	·309	...	...	60

19. It will be seen that for whole-life assurances the ratio has a very nearly constant value of about 30 per-cent, except in the case of entrants at the youngest ages, and that in the case of endowment assurances the ratios do not differ widely from that value. The results of dividing expenditure in proportion to the

value of E will therefore differ only slightly from the results obtained by dividing it in proportion to ϕ.

From this it follows that the results which are obtained by the adoption of the principles of paragraph 8 are very close to those which would be obtained by the use of a more elaborate system, according to which an attempt is made to apportion the expenditure among the various policies, and the miscellaneous profits are divided in proportion to the loading.

THE AVERAGE RESULTS OF FIRST-CLASS OFFICES.

20. It has been stated above that in the case of an average first-class company the amount of profit or loss coming under heading (*c*) of paragraph 14 will be comparatively small, and that the result of the profit distribution is practically to return the whole of the loading calculated according to Sprague's "Select Tables" at the valuation rate of interest, in addition to the whole of the interest profit. This can be shown as follows: assuming the office to earn interest at the rate of $4\frac{1}{8}$ per-cent per annum, and to declare quinquennial profits representing on the average a compound reversionary bonus at the rate of £1. 10*s*. per-cent per annum, with an interim bonus at the same rate, the gross premium should, on the hypothesis that the whole of the loading and surplus interest is returned in the shape of profits, be approximately equal to the net premium (calculated on the basis of select tables, with $4\frac{1}{8}$ per-cent interest) for the sum assured plus a compound reversionary bonus at £1. 10*s*. per-cent per annum. The following table shows how nearly this is the case:

TABLE C.

Age at Entry	Whole-Life Assurance		Endowment Assurance at 60		Age at Entry
	Gross Premium	Net Premium on basis of Select Tables, with $4\frac{1}{8}$ per-cent Interest, for Sum Assured and Compound Reversionary Bonus of £1. 10*s*. per-cent per annum	Gross Premium	Net Premium on basis of Select Tables, with $4\frac{1}{8}$ per-cent Interest, for Sum Assured and Compound Reversionary Bonus of £1. 10*s*. per-cent per annum	
20	1·95	2·08	2·45	2·56	20
25	2·18	2·24	2·85	2·87	25
30	2·45	2·49	3·40	3·35	30
35	2·80	2·81	4·15	4·03	35
40	3·22	3·23	5·30	5·04	40
45	3·80	3·77	7·20	6·73	45
50	4·55	4·47	11·05	10·01	50
55	5·52	5·38	...	...	55
60	6·90	6·62	...	...	60

21. Dealing first with the whole-life assurances, the agreement is seen to be very close in the case of assurances effected at ages from 30 to 45, the ages at which policies are principally effected, but the younger entrants obtain rather more, and the older entrants rather less than the whole of the loading. In the case of endowment assurances it would seem that the effect of the compound reversionary bonus at 30*s.* per-cent per annum is to return rather less than the whole loading, so that, from the present point of view, endowment assurances appear to be treated less favourably than whole-life assurances.

Comparison of the Results of the Premium System and of the Loading System.

22. From the rates given in Table A the following table has been deduced, showing the loading and ratio of loading to gross premiums on four different assumptions as to the basis of calculation of the net or pure premiums:

Table D.—1. *Whole-Life Assurances.*

Age at Entry	Net Premium P_x	Loading ϕ_x	Ratio of Loading to Gross Premium $\phi_x \div \pi_x$	Net Premium P_x	Loading ϕ_x	Ratio of Loading to Gross Premium $\phi_x \div \pi_x$	Age at Entry
	H^M 3 per-cent			H^M 3½ per-cent			
20	1·427	·523	·268	1·330	·620	·318	20
25	1·625	·550	·253	1·521	·654	·301	25
30	1·880	·570	·233	1·769	·681	·278	30
35	2·193	·607	·217	2·076	·724	·258	35
40	2·589	·636	·197	2·465	·760	·236	40
45	3·114	·686	·181	2·985	·815	·214	45
50	3·801	·749	·165	3·667	·883	·194	50
55	4·725	·800	·145	4·588	·937	·170	55
60	5·987	·913	·132	5·848	1·052	·152	60
	Sprague's Select Table 3 per-ct.			Sprague's Select Table 3½ per-ct.			
20	1·564	·386	·198	1·471	·479	·246	20
25	1·703	·472	·217	1·600	·575	·264	25
30	1·925	·525	·214	1·814	·636	·260	30
35	2·218	·582	·208	2·100	·700	·250	35
40	2·602	·623	·193	2·477	·748	·232	40
45	3·106	·694	·183	2·974	·826	·217	45
50	3·755	·795	·175	3·618	·932	·205	50
55	4·635	·890	·161	4·492	1·033	·187	55
60	5·826	1·074	·156	5·681	1·219	·177	60

TABLE D—(*continued*).

2. *Endowment Assurances at* 60.

Age at Entry	Net Premium $P_{x:\overline{60-x}}$	Loading $\phi_{x:\overline{60-x}}$	Ratio of Loading to Gross Premium $\phi_{x:\overline{60-x}} \div \pi_{x:\overline{60-x}}$	Net Premium $P_{x:\overline{60-x}}$	Loading $\phi_{x:\overline{60-x}}$	Ratio of Loading to Gross Premium $\phi_{x:\overline{60-x}} \div \pi_{x:\overline{60-x}}$	Age at Entry
	H^{M} 3 per-cent			H^{M} $3\frac{1}{2}$ per-cent			
20	1·864	·586	·239	1·733	·717	·293	20
25	2·212	·638	·224	2·070	·780	·274	25
30	2·694	·706	·208	2·540	·860	·253	30
35	3·366	·784	·189	3·199	·951	·229	35
40	4·373	·927	·175	4·190	1·110	·209	40
45	6·060	1·140	·158	5·859	1·341	·186	45
50	9·389	1·661	·150	9·165	1·885	·171	50
	Sprague's Select Table 3 per-ct.			Sprague's Select Table $3\frac{1}{2}$ per-ct.			
20	1·997	·453	·185	1·871	·579	·237	20
25	2·281	·569	·200	2·141	·709	·249	25
30	2·727	·673	·198	2·571	·829	·244	30
35	3·371	·779	·188	3·203	·947	·228	35
40	4·357	·943	·178	4·173	1·127	·213	40
45	6·003	1·197	·166	5·801	1·399	·194	45
50	9·253	1·797	·163	9·028	2·022	·183	50

On consideration of the results of this table we shall arrive at the following conclusions:

(1) Assuming that the average whole-life premium in respect of policies in force at the date of distribution is about that for age 37,* and the average endowment assurance premium about that for a 25 year term, the percentage of loading to gross

* This is not the same as the average premium in respect of new policies *effected*. It will be found that as the average duration of the policies increases the average premium decreases, because the entrants at the higher ages die off more rapidly than the younger entrants. The following figures, based on Mr. King's hypothetical office (*J.I.A.*, xx, 233) will illustrate this point:

Age of Office Years	Number of Policies existing	Total Net Premium H^{M} 3 per-cent	Average Net Premium per-cent	Age to which Average Premium corresponds
0	127,471	3,092	2·425	38 +
25	1,895,857	44,957	2·371	37 −
50	2,515,624	57,498	2·286	36 +

premium is in the aggregate very nearly the same whether the H^M or select net premium be employed, but the average percentage is smaller for endowment assurances than for whole-life assurances. From this it follows that

(2) As between endowment assurances as a *class* and whole-life assurances as a *class*, approximately the same result would be obtained by the H^M loading plan and the select loading plan, and both would give a smaller proportion of surplus to the endowment assurances than would be given by the premium system.

(3) The ratio $\phi : \pi$ decreases as the age at entry increases, and therefore the loading system gives a larger proportion of the cash surplus divisible among whole-life assurances to assurances effected on young lives, and also gives a larger proportion of the cash surplus divisible among endowment assurances to policies effected for a long term, than would respectively be allotted under the premium system.

(4) The features mentioned under heading (3) are not so strongly marked when the Select Tables are used to determine the net premium as when the H^M Table is used.

23. The above conclusions may be exhibited in a clearer light if we assume a representative rate of allotment, and thence determine the actual amount which would be allocated (in addition to the interest profit) under the three different plans, to policies effected at various ages. The figures thus obtained will also serve as the basis of subsequent calculations, showing the amount of bonus addition which would be produced by the respective systems. Consistently with what has gone before it will be assumed:

(1) That the office premiums are those given in Table A.

(2) That the valuations are made on the basis of the combined H^M and $H^{M(5)}$ Tables, with interest at 3 per-cent.

(3) That the average value of the ratio $\phi:\pi$ will be as follows: whole-life H^M ·207; ditto, Select, ·200; endowment assurances, H^M ·189; ditto, Select, ·188.

(4) That the participating whole-life assurances are to the participating endowment assurances in the ratio of 10 to 1.

(5) That the rate of interest earned is $4\frac{1}{8}$ per-cent per annum.

(6) That the distributions of profits are made annually, and that the total amount divided at the end of each year is equivalent to the whole of the interest profit on the reserve, plus the whole of the loading calculated according to the Select Tables at 3 per-cent interest.

24. From these data we shall obtain the figures given in Table E.

TABLE E.—*Showing the Cash Allotment to Policies effected at various ages, excluding the Interest Profits on Reserves.*

Age at Entry	Whole-Life Assurances			Endowment Assurances at 60			Age at Entry
	Premium System	H^M Loading System	Select Loading System	Premium System	H^M Loading System	Select Loading System	
20	·388	·507	·386	·488	·568	·453	20
25	·433	·533	·472	·567	·618	·569	25
30	·488	·552	·525	·677	·684	·673	30
35	·557	·588	·582	·826	·760	·779	35
40	·642	·616	·623	1·055	·898	·943	40
45	·756	·665	·694	1·433	1·105	1·197	45
50	·906	·726	·795	2·199	1·609	1·797	50
55	1·100	·775	·890	...	...	...	55
60	1·373	·885	1·074	...	...	...	60

25. It will be seen that the numerical results of Table E confirm the general conclusions of paragraph 22, and that, speaking generally, the results of the select loading system lie between those of the premium system and the H^M loading system. Comparing the results, age by age, it would appear that under the assumed conditions the premium system gives too

small a proportion of surplus to policies subject to low rates of premium, and too large a proportion to policies subject to high rates of premium, while in the case of the H^M loading plan the conditions are reversed.

26. It has been laid down as one of the criteria of a good method of distribution of surplus that assuming the real profit-earning power of a company to remain unaltered, no considerable change in the amount of surplus allotted to policies effected at various ages and of different durations should be produced merely by a change in the relative proportions of the new business effected at different ages and under different tables.* This must, perhaps, be regarded as an ideal result, which it is impossible in practice to fully secure, but it nevertheless gives a good standard by which to judge the comparative merits of different methods. Applying this test to the premium system it is submitted that, under the supposed conditions, which have been chosen to represent as nearly as possible those which obtain in an average first-class company, the method does, to a certain extent, provide the bonuses allotted to policies effected on old lives and to endowment assurances having a short term to run out of the profit contributed by the younger entrants, and that, therefore, an increase in the proportion of new assurances belonging to the former classes will have the effect of diminishing the profits on policies of the latter class. The question is one of great and growing importance, having regard to the rapid increase, both absolute and proportional, which is taking place in the amount of assurances effected under the endowment assurance system, and more especially in view of the fact that such assurances are now frequently taken out for short terms by way of investment.

27. It seems hardly necessary to again point out—although to avoid misunderstanding it may be desirable to do so—that the above remarks will have to be very considerably modified if the premiums at the higher ages are greater than those given in Table A.

* See some very interesting remarks by Mr. T. G. C. Browne on this subject (*J.I.A.*, xxx. 148).

DISTRIBUTION AT INTERVALS GREATER THAN ONE YEAR.

28. For the sake of convenience it has been assumed up to this point that the distribution of profits is made annually, and in theory this seems the most natural course to adopt. In practice, however, the valuation and division of surplus are most commonly made quinquennially, and it becomes desirable to consider what effect this may have upon the method of distribution. Probably the chief reasons for the adoption of quinquennial instead of annual valuations are the following:

(1) By dealing with the profits earned over a period of five years we are likely to secure much more stability in the results of successive distributions than by dealing with the profits of individual years, which may show considerable fluctuations.

(2) The labour and expense attending a valuation and the consequent allotment of profits are so considerable that it is desirable, for the sake of practical convenience, to avoid having an annual distribution.

29. It seems proper, therefore, that the system of allotment adopted at the quinquennial distributions should be so adjusted as to produce results as nearly as possible equivalent to those which would be obtained if the profit were earned at a uniform rate during the quinquennium, and were actually distributed annually. This condition will be fulfilled if the present value at the beginning of any quinquennium of the profit to be allotted at the expiration thereof, plus the value of the interim bonus payable in the event of death during the quinquennium, be equal to the value of the profit which would be allotted in respect of the same period under the system of annual distribution. We shall thus have an equation of condition between the amounts of the quinquennial bonuses and the interim bonuses respectively.

30. There will be in practice three principal cases to consider:

(*a*) If no interim bonuses be allowed it will be proper to accumulate the annual cash allotment at interest, with benefit of survivorship, in order to obtain the

amount of cash surplus to be allotted at the end of the quinquennium.

(*b*) If the interim bonus allowed in case of death be the amount of *cash surplus* accrued since the last quinquennial distribution, the cash surplus to be allotted at the end of the quinquennium to those who survive will be found by accumulating the amount of cash surplus at interest only.

(*c*) If the interim bonus be the amount of bonus *addition* which would have accrued since the last quinquennial distribution, if the profits had been allotted and converted into bonus additions annually, the formulas for the periodical and interim bonuses become somewhat complicated, and they are therefore investigated in detail in Appendix A.

31. The principles of paragraph 29 will be most completely carried out by the adoption of method (*c*), which has accordingly been used in obtaining specimens of the amounts of the bonus additions which would respectively be allotted by the premium system, the H^M loading system and the select loading system, on the assumption that the annual cash surplus, in addition to the surplus interest, is that shown in Table E.

32. The specimen bonuses thus obtained are exhibited in the tables which are included in Appendix A. Perhaps the most noteworthy feature of the figures there given is the remarkable agreement between the results obtained, in respect of whole-life assurances, by the use of the H^M loading system and the uniform compound reversionary bonus system. The select loading plan, as would have been anticipated from the reasoning of paragraphs 21 and 25, produces bonus additions which are smaller for the younger ages at entry and larger for the higher ages at entry than those which would be allotted according to the compound reversionary bonus plan. The tables thus confirm the results of previous investigations, which have shown that if the rate of interest realized be about 1 per-cent per annum in excess of that assumed in the valuation, the uniform bonus plan will give very satisfactory results, provided that the premiums be suitably adjusted; and it would appear that the requisite adjustment would be obtained by slightly increasing the premiums of Table A for the younger ages and decreasing them at the older ages.

33. Similar remarks will apply in the case of endowment assurances, but the tables do not confirm the view, which is held by some actuaries, that endowment assurances, as a class, are not entitled to as high a rate of bonus as that allotted to whole-life assurances.

Expressions for the Present Value of Future Allotments of Surplus.

34. It will be useful, for many purposes, to have expressions for the *present value* at any time of the surplus to be allotted in the future, on the assumption of any definite scale of cash allotment; such expressions will, for example, be of great assistance in comparing the average results obtained under different systems of distribution. In discussing this question, it will be assumed, for convenience, that the distributions are either made annually or are so adjusted as to produce results equivalent to those derived from an annual allotment.

35. There is no difficulty in obtaining the present value of that portion of the surplus which is divided in proportion to the premiums or the loading. If a stationary rate of allotment be assumed, as for example $K\phi$ or $K\pi$ per annum, the value at any time will be $(K\phi)a$ or $(K\pi)a$, where a represents the annuity-value according to the table which is being used in the calculation, and if K be assumed to vary we shall have to take the value of a corresponding varying annuity which may be represented by the expression $\phi(Va)$ or $\pi(Va)$.

36. The formulas for the present value of the profit from surplus interest are much more complicated, and they might appear, at first sight, to be practicably irreducible to a simple form. Nevertheless, it will be found on further investigation that the formulas for the present value (if the calculation be made on the basis of the "experience" rate of interest) reduce to the following simple and interesting form:

> [Valuation Reserve − Reserve according to experience rate of interest] + [present value of the difference between the premium used in the valuation and the corresponding premium according to the experience rate of interest].

This result is perfectly general, and when once stated will no doubt appear to many actuaries to be sufficiently obvious from general considerations. Nevertheless it appears to be desirable to demonstrate the formula algebraically, and the analysis, which will be found to be of an interesting character, is therefore given in detail in Appendix B, together with some further developments which may in some cases be practically useful.

Concluding Remarks.

37. The results given in the two preceding paragraphs might, it is thought, be of considerable use in the investigation of many important and interesting questions, such, for example, as the following: What is the effect of the system which divides the surplus in proportion to the premiums paid? How would the position of existing policyholders be affected by a change from such a system to one based upon Dr. Sprague's principles? What is the effect of a reduction in the rate of interest assumed in the valuation? It had originally been intended to touch upon these questions, but this paper has already extended to such length that any further discussion must be reserved for another occasion. The writer will therefore conclude by earnestly appealing to the senior members of the Institute, whose ripe experience peculiarly qualifies them to assist those who are studying one of the most important and difficult branches of actuarial science, to give the younger members the benefit of their views, not only upon the particular questions referred to in the preceding notes, but also upon the general subject of the distribution of the surplus of a life assurance company.

APPENDIX A.

Let us assume that the surplus interest earned is at the rate ρ per annum, and that the annual cash surplus allotted at the end of each year, in addition to the interest profit, is κ per unit assured. Further, let

$V' = V + P$ = reserve in respect of sum assured alone immediately after payment of the premium.

B = bonus additions existing at the beginning of any quinquennium.

A = assurance value on valuation basis,

and let the subscripts 0, 1, 2 . . . . respectively denote that the functions are calculated at the beginning of the quinquennium, one year later and so on. Then on the hypothesis that the cash surplus is allotted annually and is converted into reversionary bonus according to the valuation basis, the following table shows the total new bonuses (*i.e.*, excluding the bonus B) at the end of the 1st, 2nd . . . . 5th year of the quinquennium.

Year	BONUS ADDITION ARISING FROM (1) Surplus Interest on Policy Reserve	(2) Annual Cash Allotment of κ	(3) Surplus Interest on Reserve in respect of Bonus of B
	increased in each case by the Bonus arising from Surplus Interest on Profit from the same source in previous years of the quinquennium		
1	$\rho\dfrac{V'_0}{A_1}$	$\dfrac{\kappa}{A_1}$	$\rho B\dfrac{A_0}{A_1}$
2	$\rho\dfrac{V'_0}{A_1}\left(1+\rho\dfrac{A_1}{A_2}\right)+\rho\dfrac{V'_1}{A_2}$	$\dfrac{\kappa}{A_1}\left(1+\rho\dfrac{A_1}{A_2}\right)+\dfrac{\kappa}{A_2}$	$\rho B\dfrac{A_0}{A_1}\left(1+\rho\dfrac{A_1}{A_2}\right)+\rho B\dfrac{A_1}{A_2}$
⋮	⋮	⋮	⋮
5 (putting $\dfrac{A_n}{A_{n+1}}=X_n$ &c.)	$\rho\left[\dfrac{V'_0}{A_1}X_1X_2X_3X_4+\dfrac{V'_1}{A_2}X_2X_3X_4+\dfrac{V'_2}{A_3}X_3X_4+\dfrac{V'_3}{A_4}X_4+\dfrac{V'_4}{A_5}\right]$	$\kappa\left[\dfrac{X_1X_2X_3X_4}{A_1}+\dfrac{X_2X_3X_4}{A_2}+\dfrac{X_3X_4}{A_3}+\dfrac{X_4}{A_4}+\dfrac{1}{A_5}\right]$	$\rho B\left[\dfrac{A_0}{A_1}X_1X_2X_3X_4+\dfrac{A_1}{A_2}X_2X_3X_4+\dfrac{A_2}{A_3}X_3X_4+\dfrac{A_3}{A_4}X_4+\dfrac{A_4}{A_5}\right]$

Since ρ is a small quantity and $\frac{A_x}{A_{x+1}}$ is very nearly equal to unity, we may, for practical purposes, substitute $1+\rho$ for X throughout, and if we further substitute five times the central value for the sum of five values, the results at the end of the fifth year may be approximately represented by the following formulas:

$$\left.\begin{array}{llll} \text{Column (1)} & . \quad . \quad . & \dfrac{5(1+\rho)^2}{A_3}\rho V'_2 \\[2ex] \quad ,, \quad (2) & . \quad . \quad . & \dfrac{5(1+\rho)^2}{A_3}\kappa \\[2ex] \quad ,, \quad (3) & . \quad . \quad . & \dfrac{5(1+\rho)^2}{A_3}\rho B A_2 \end{array}\right\}$$

These formulas have been employed, in conjunction with the hypothetical rates of allotment referred to in paragraph 24 as the basis of calculation of the following tables showing the amounts of reversionary bonus added to policies of 1,000 according to the premium system, the H^M loading system, and the select loading system respectively. The tables are so arranged as to exhibit separately the bonus additions arising from each of the three sources of surplus (1, 2 and 3) referred to in the preceding tabular statement, and they also show the total bonuses and the results which would obtain if each quinquennial bonus were surrendered at the date of allotment, in which case there would be no profit under heading (3). For comparison, there has been added at the foot of each table a statement of the effect of a compound reversionary bonus of £1. 10*s*. 0*d*. per-cent per annum.

In forming these tables for whole-life assurances, A has been taken from Sprague's Select Tables, V' has been calculated on the basis of the H^M Table for the first quinquennium and the combined H^M and $H^{M(5)}$ thereafter. In the case of endowment assurances, both A and V' have, for convenience, been taken from the H^M Table throughout.

WHOLE-LIFE ASSURANCES—AGE AT ENTRY 20.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45	46–50	51–55
Premium System	(1)	4·9	14·4	19·5	24·9	29·8	34·6	38·8	42·5	45·7	48·5	50·8
	(2)	54·2	50·9	46·9	43·0	39·4	36·0	33·0	30·3	28·1	26·2	24·6
	(1) + (2)	59·1	65·3	66·4	67·9	69·2	70·6	71·8	72·8	73·8	74·7	75·4
	(3)	...	3·4	7·2	11·4	15·9	20·7	25·9	31·4	37·4	43·7	50·6
	(1) + (2) + (3)	59·1	68·7	73·6	79·3	85·1	91·3	97·7	104·2	111·2	118·4	126·0
	Total Bonuses	59·1	127·8	201·4	280·7	365·8	457·1	554·8	659·0	770·2	888·6	1014·6
H^M Loading System	(1)	4·9	14·4	19·5	24·9	29·8	34·6	38·8	42·5	45·7	48·5	50·8
	(2)	70·8	66·5	61·3	56·2	51·5	47·0	43·1	39·6	36·7	34·2	32·1
	(1) + (2)	75·7	80·9	80·8	81·1	81·3	81·6	81·9	82·1	82·4	82·7	82·9
	(3)	...	4·3	9·1	14·2	19·6	25·3	31·3	37·8	44·6	51·9	59·6
	(1) + (2) + (3)	75·7	85·2	89·9	95·3	100·9	106·9	113·2	119·9	127·0	134·6	142·5
	Total Bonuses	75·7	160·9	250·8	346·1	447·0	553·9	667·1	787·0	914·0	1048·6	1191·1
Select Loading System	(1)	4·9	14·4	19·5	24·9	29·8	34·6	38·8	42·5	45·7	48·5	50·8
	(2)	53·9	50·6	46·7	42·8	39·2	35·8	32·8	30·2	27·9	26·0	24·5
	(1) + (2)	58·8	65·0	66·2	67·7	69·0	70·4	71·6	72·7	73·6	74·5	75·3
	(3)	...	3·3	7·2	11·3	15·8	20·6	25·7	31·3	37·2	43·6	50·4
	(1) + (2) + (3)	58·8	68·3	73·4	79·0	84·8	91·0	97·3	104·0	110·8	118·1	125·7
	Total Bonuses	58·8	127·1	200·5	279·5	364·3	455·3	552·6	656·6	767·4	885·5	1011·2
Compound Reversionary Bonus of 30*s.* per-cent per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115·7	124·5	133·7	143·8	154·6
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0	783·5	917·2	1061·0	1215·6

WHOLE-LIFE ASSURANCES—AGE AT ENTRY 30.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45
Premium System	(1)	5·8	14·9	21·4	27·6	33·1	38·0	42·3	45·8	48·9
	(2)	59·0	54·1	49·6	45·3	41·5	38·2	35·3	32·9	30·9
	(1) + (2)	64·8	69·0	71·0	72·9	74·6	76·2	77·6	78·7	79·8
	(3)	...	3·7	7·8	12·2	17·0	22·2	27·9	33·9	40·3
	(1) + (2) + (3)	64·8	72·7	78·8	85·1	91·6	98·4	105·5	112·6	120·1
	Total Bonuses	64·8	137·5	216·3	301·4	393·0	491·4	596·9	709·5	829·6
H^M Loading System	(1)	5·8	14·9	21·4	27·6	33·1	38·0	42·3	45·8	48·9
	(2)	66·7	61·2	56·1	51·2	46·9	43·2	39·9	37·2	35·0
	(1) + (2)	72·5	76·1	77·5	78·8	80·0	81·2	82·2	83·0	83·9
	(3)	...	4·1	8·6	13·5	18·7	24·3	30·3	36·8	43·6
	(1) + (2) + (3)	72·5	80·2	86·1	92·3	98·7	105·5	112·5	119·8	127·5
	Total Bonuses	72·5	152·7	238·8	331·1	429·8	535·3	647·8	767·6	895·1
Select Loading System	(1)	5·8	14·9	21·4	27·6	33·1	38·0	42·3	45·8	48·9
	(2)	63·5	58·2	53·3	48·7	44·6	41·1	38·0	35·4	33·3
	(1) + (2)	69·3	73·1	74·7	76·3	77·7	79·1	80·3	81·2	82·2
	(3)	...	3·9	8·3	13·0	18·0	23·4	29·3	35·6	42·3
	(1) + (2) + (3)	69·3	77·0	83·0	89·3	95·7	102·5	109·6	116·8	121·5
	Total Bonuses	69·3	146·3	229·3	318·6	414·3	516·8	626·4	743·2	867·7
Compound Reversionary Bonus of 30s. per-cent per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115·7	124·5	133·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0	783·5	917·2

WHOLE-LIFE ASSURANCES—AGE AT ENTRY 40.

Bonus Additions to a Policy of 1,000 *in respect of the under-mentioned Years of Assurance.*

System of Allotment	Profit arising under Heading	1–5	6–10	11–15	16–20	21–25	26–30	31–35
Premium System	(1)	6·8	16·7	24·2	31·0	36·8	41·7	45·9
	(2)	65·2	59·5	54·6	50·2	46·4	43·3	40·7
	(1) + (2)	72·0	76·2	78·8	81·2	83·2	85·0	86·6
	(3)	...	4·1	8·6	13·6	19·0	24·8	31·1
	(1) + (2) + (3)	72·0	80·3	87·4	94·8	102·2	109·8	117·7
	Total Bonuses	72·0	152·3	239·7	334·5	436·7	546·5	664·2
H^M Loading System	(1)	6·8	16·7	24·2	31·0	36·8	41·7	45·9
	(2)	62·6	57·1	52·4	48·2	44·5	41·6	39·0
	(1) + (2)	69·4	73·8	76·6	79·2	81·3	83·3	84·9
	(3)	...	3·9	8·3	13·1	18·4	24·1	30·2
	(1) + (2) + (3)	69·4	77·7	84·9	92·3	99·7	107·4	115·1
	Total Bonuses	69·4	147·1	232·0	324·3	424·0	531·4	646·5
Select Loading System	(1)	6·8	16·7	24·2	31·0	36·8	41·7	45·9
	(2)	63·3	57·8	53·0	48·7	45·1	42·0	39·5
	(1) + (2)	70·1	74·5	77·2	79·7	81·9	83·7	85·4
	(3)	...	4·0	8·4	13·3	18·5	24·3	30·5
	(1) + (2) + (3)	70·1	78·5	85·6	93·0	100·4	108·0	115·9
	Total Bonuses	70·1	148·6	234·2	327·2	427·6	535·6	651·5
Compound Reversionary Bonus of 30*s.* per-cent per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0

WHOLE-LIFE ASSURANCES—*Ages at Entry* 50 *and* 60.

Bonus Additions to a Policy of 1,000 *in respect of the under-mentioned Years of Assurance.*

System of Allotment	Profit arising under Heading	Age at Entry 50: 1–5	6–10	11–15	16–20	21–25	Age at Entry 60: 1–5	6–10	11–15
Premium System	(1)	8·2	19·0	27·6	34·7	40·8	10·2	21·9	31·5
	(2)	77·2	70·8	65·5	61·1	57·4	99·3	92·6	87·0
	(1) + (2)	85·4	89·8	93·1	95·8	98·2	109·5	114·5	118·5
	(3)	...	4·8	10·2	16·1	22·5	...	6·2	13·1
	(1) + (2) + (3)	85·4	94·6	103·3	111·9	120·7	109·5	120·7	131·6
	Total Bonuses	85·4	180·0	283·3	395·2	515·9	109·5	230·2	361·8
H^M Loading System	(1)	8·2	19·0	27·6	34·7	40·8	10·2	21·9	31·5
	(2)	61·8	56·8	52·5	49·0	46·0	64·0	59·7	56·1
	(1) + (2)	70·0	75·8	80·1	83·7	86·8	74·2	81·6	87·6
	(3)	...	4·0	8·5	13·5	19·1	...	4·2	9·1
	(1) + (2) + (3)	70·0	79·8	88·6	97·2	105·9	74·2	85·8	96·7
	Total Bonuses	70·0	149·8	238·4	335·6	441·5	74·2	160·0	256·7
Select Loading System	(1)	8·2	19·0	27·6	34·7	40·8	10·2	21·9	31·5
	(2)	67·7	62·2	57·5	53·6	50·4	77·7	72·5	68·0
	(1) + (2)	75·9	81·2	85·1	88·3	91·2	87·9	94·4	99·5
	(3)	...	4·3	9·1	14·5	20·4	...	5·0	10·6
	(1) + (2) + (3)	75·9	85·5	94·2	102·8	111·6	87·9	99·4	110·1
	Total Bonuses	75·9	161·4	255·6	358·4	470·0	87·9	187·3	297·4
Compound Reversionary Bonus of 30s. per-cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	75·0	80·6	86·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	75·0	155·6	242·3

ENDOWMENT ASSURANCES AT 60—AGE AT ENTRY 20.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40
Premium System	(1)	6·2	14·7	22·6	29·9	36·6	42·9	48·6	53·8
	(2)	60·2	54·2	48·6	43·5	38·8	34·3	30·3	26·5
	(1) + (2)	66·4	68·9	71·2	73·4	75·4	77·2	78·9	80·3
	(3)	...	3·7	7·8	12·3	17·1	22·2	27·8	33·7
	(1) + (2) + (3)	66·4	72·6	79·0	85·7	92·5	99·4	106·7	114·0
	Total Bonuses	66·4	139·0	218·0	303·7	396·2	495·6	602·3	716·3
H^M Loading System	(1)	6·2	14·7	22·6	29·9	36·6	42·9	48·6	53·8
	(2)	70·1	63·1	56·6	50·6	45·1	40·0	35·2	30·8
	(1) + (2)	76·3	77·8	79·2	80·5	81·7	82·9	83·8	84·6
	(3)	...	4·3	8·9	13·9	19·2	24·8	30·8	37·1
	(1) + (2) + (3)	76·3	82·1	88·1	94·4	100·9	107·7	114·6	121·7
	Total Bonuses	76·3	158·4	246·5	340·9	441·8	549·5	664·1	785·8
Select Loading System	(1)	6·2	14·7	22·6	29·9	36·6	42·9	48·6	53·8
	(2)	55·9	50·3	45·2	40·4	36·0	31·9	28·1	24·6
	(1) + (2)	62·1	65·0	67·8	70·3	72·6	74·8	76·7	78·4
	(3)	...	3·5	7·4	11·6	16·2	21·1	26·6	32·2
	(1) + (2) + (3)	62·1	68·5	75·2	81·9	88·8	95·9	103·3	110·6
	Total Bonuses	62·1	130·6	205·8	287·7	376·5	472·4	575·7	686·3
Compound Reversionary Bonus of 30*s*. per-cent per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115·7	124·5
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0	783·5

ENDOWMENT ASSURANCES AT 60—AGE AT ENTRY 30.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1–5	6–10	11–15	16–20	21–25	26–30
Premium System	(1)	7·6	18·2	27·9	37·0	45·2	52·9
	(2)	67·5	60·4	53·8	47·6	42·0	36·7
	(1) + (2)	75·1	78·6	81·7	84·6	87·2	89·6
	(3)	...	4·2	8·9	13·9	19·5	25·4
	(1) + (2) + (3)	75·1	82·8	90·6	98·5	106·7	115·0
	Total Bonuses	75·1	157·9	248·5	347·0	453·7	568·7
H^{M} Loading System	(1)	7·6	18·2	27·9	37·0	45·2	52·9
	(2)	68·2	61·0	54·3	48·2	42·5	37·1
	(1) + (2)	75·8	79·2	82·2	85·2	87·7	90·0
	(3)	...	4·3	8·9	14·1	19·6	25·6
	(1) + (2) + (3)	75·8	83·5	91·1	99·3	107·3	115·6
	Total Bonuses	75·8	159·3	250·4	349·7	457·0	572·6
Select Loading System	(1)	7·6	18·2	27·9	37·0	45·2	52·9
	(2)	67·1	60·0	53·4	47·4	41·8	36·5
	(1) + (2)	74·7	78·2	81·3	84·4	87·0	89·4
	(3)	...	4·2	8·8	13·9	19·4	25·3
	(1) + (2) + (3)	74·7	82·4	90·1	98·3	106·4	114·7
	Total Bonuses	74·7	157·1	247·2	345·5	451·9	566·6
Compound Reversionary Bonus of 30*s*. per-cent per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3

Endowment Assurances at 60—*Ages at Entry* 40 *and* 50.

Bonus Additions to a Policy of 1,000 *in respect of the under-mentioned Years of Assurance.*

System of Allotment	Profit arising under Heading	Age at Entry 40				Age at Entry 50	
		1–5	6–10	11–15	16–20	1–5	6–10
Premium System	(1)	10·3	25·0	38·4	51·0	18·2	45·4
	(2)	83·8	74·3	65·5	57·2	136·5	119·2
	(1) + (2)	94·1	99·3	103·9	108·2	154·7	164·6
	(3)	...	5·3	11·1	17·4	...	8·7
	(1) + (2) + (3)	94·1	104·6	115·0	125·6	154·7	173·3
	Total Bonuses	94·1	198·7	313·7	439·3	154·7	328·0
H^M Loading System	(1)	10·3	25·0	38·4	51·0	18·2	45·4
	(2)	71·3	63·2	55·7	48·7	99·9	87·2
	(1) + (2)	81·6	88·2	94·1	99·7	118·1	132·6
	(3)	...	4·6	9·8	15·5	...	6·6
	(1) + (2) + (3)	81·6	92·8	103·9	115·2	118·1	139·2
	Total Bonuses	81·6	174·4	278·3	393·5	118·1	257·3
Select Loading System	(1)	10·3	25·0	38·4	51·0	18·2	45·4
	(2)	74·9	66·4	58·5	51·1	111·5	97·4
	(1) + (2)	85·2	91·4	96·9	102·1	129·7	142·8
	(3)	...	4·8	10·2	16·1	...	7·3
	(1) + (2) + (3)	85·2	96·2	107·1	118·2	129·7	150·1
	Total Bonuses	85·2	181·4	288·5	406·7	129·7	279·8
Compound Reversionary Bonus of 30*s.* per-cent per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	75·0	80·6
	Total Bonuses	75·0	155·6	242·3	335·5	75·0	155·6

APPENDIX B.

Demonstration of the Expression given in paragraph 36 *for the Present Value of the Profit from Surplus Interest.*

Taking first the simple case of a whole-life policy on a life aged x at entry, let

π = the valuation premium, which need not necessarily be the pure or net premium.

i = the valuation rate of interest.

${}_n\mathrm{V}_x$ = the reserve at the expiration of n years.

Then ${}_n\mathrm{V}_x+\pi$ will be the reserve at the commencement of the $(n+1)$th year immediately after payment of the premium, and will therefore be the amount on which the surplus interest is earned during the year.

Further, let the same symbols when accented denote quantities calculated according to the experience rate of interest, i' say.

Now ${}_n\mathrm{V}_x+\pi=1-(1+a_{x+n})(\pi+d)+\pi=v-a_{x+n}(\pi+d)$.

Thus, at the beginning of the $\overline{n+1}$ th year, the present value, calculated at the experience rate i', of the profit for the $\overline{n+1}$ th year will be

$$v'(i'-i)\left[v-a_{x+n}(\pi+d)\right]$$

and the present value of the profit for the $(n+2)$th year

$$v'^2(i'-i)\left[v-a_{x+n+1}(\pi+d)\right]{}_1p_x$$

and so on.

Summing, and introducing obvious abbreviations in the notation, we have

$$\Sigma=v'(i'-i)\left[v+v'v_1p+v'^2v_2p+\ .\ .\ \ -(\pi+d)(a+v'{}_1pa_1+v'^2{}_2pa_2+\ .\ .\ .)\right]$$

$$=v'(i'-i)\left[(1+a')v-(\pi+d)\times\mathrm{B},\ \text{say}\right]\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (1)$$

Now
$$\begin{aligned}B = & v_1p + v^2{}_2p + v^3{}_3p + \ldots \\ & + v'v_2p + v'v^2{}_3p + \ldots \\ & + v'^2v_3p + \ldots \\ & \ldots\ldots\ldots\ldots \\ = & v_1p + {}_2p(v'v + v^2) + {}_3p(v^3 + v^2v' + vv'^2) + \ldots\end{aligned}$$

which will be found to reduce to

$$\frac{v}{v-v'}(a-a') \quad \ldots\ldots\ldots \quad (2)$$

Substituting in (1), we obtain

$$\begin{aligned}\Sigma &= v'(i'-i)\left[v(1+a') - (\pi+d)(a-a')\frac{v}{v-v'}\right] \\ &= vv'(i'-i)\left[1+a' - (\pi+d)(a-a')\frac{1}{v-v'}\right] \quad \ldots \quad (3)\end{aligned}$$

Now $\quad vv'(i'-i) = vv'(\overline{1+i'} - \overline{1+i}) = v - v' = d' - d.$

Therefore expression (3) reduces to

$$\begin{aligned}&(1+a')(d'-d) - (\pi+d)(a-a') \\ &= (1+a')(\pi+d') - (1+a)(\pi+d) \quad \ldots\ldots \quad (4) \\ &= A - A' + [(1+a) - (1+a')]\pi \\ &= {}_nV_x - {}_nV'_x + (1+a'_{x+n})(\pi - \pi') \quad \ldots\ldots \quad (5)\end{aligned}$$

where $\;{}_nV'_x = 1 - (1+a'_{x+n})(\pi' + d')$.

Formula (5) expresses the present value, calculated at the *experience rate of interest*, of the profit from excess interest on the reserves, and from it the present value calculated at any other rate of interest may easily be found. Thus

$${}^iV - {}^{i'}V + {}^{i'}\mathbf{a}({}^i\pi - {}^{i'}\pi)$$

is the present value (at rate i') of interest profit at the rate of $i'-i$. Therefore

$${}^iV - {}^{i+k}V + {}^{i+k}\mathbf{a}({}^i\pi - {}^{i+k}\pi)$$

is the present value at rate $i+k$ of interest profit at rate k, and thus

$$\frac{i'-i}{k}\left[{}^{i}\mathrm{V}-{}^{i+k}\mathrm{V}+{}^{i+k}\mathbf{a}({}^{i}\pi-{}^{i+k}\pi)\right] \quad . \quad . \quad . \quad (6)$$

is the value at rate $i+k$ of interest profit at rate $i'-i$. The last expression may be written

$$(i'-i)\left[\frac{-\Delta_k\mathrm{V}-{}^{i+k}\mathbf{a}\Delta_k\pi}{k}\right]$$

(the symbol Δ_k denoting that the interval of differencing is k) where k may be positive or negative. If we put $k=0$ (*i.e.*, make the calculation at the valuation rate), the expression takes the form $\frac{0}{0}$, but its value is then obviously

$$(i'-i)\left[-\frac{d\mathrm{V}}{di}-{}^{i}\mathbf{a}\frac{d\pi}{di}\right] \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The form of expression (5) suggests that the result must be capable of considerable generalization, and we therefore proceed to investigate the most general case, introducing a varying rate of "experience" interest, and combining profit or loss from mortality with the profit or loss from interest.

Let i = the valuation rate of interest.

$i_1, i_2, i_3, \ldots\ldots$ = the rates of interest realized respectively in the 1st, 2nd, 3rd, . . . year.

$p_1, p_2, p_3, \ldots\ldots$ = the probability according to the valuation basis, that the status, if in existence at the beginning of the 1st, 2nd, 3rd, . . . year, will survive to the end of the year.*

* In the case of a compound status, $p+q$ will not necessarily be equal to unity.

$p'_1, p'_2, p'_3, \ldots$ = similar probabilities according to the rates of mortality actually experienced.

$q_1, q_2, q_3, \ldots$ = the probability, according to the valuation basis, that if the status be in existence at the beginning of the 1st, 2nd, 3rd, ... year the sum assured will become payable at the end of the year.†

$q'_1, q'_2, q'_3, \ldots$ = similar probabilities according to the rates of mortality actually experienced.

$S_1, S_2, S_3, \ldots$ = the sum assured in the event of the contingency happening in the 1st, 2nd, 3rd, ... year.

$\pi_1, \pi_2, \pi_3, \ldots$ = the premium payable at the commencement of the 1st, 2nd, 3rd, ... year.

$B_n = vq_{n+1}S_{n+1} + v^2p_{n+1}q_{n+2}S_{n+2} + \ldots$ = value of the benefit at the expiration of n years.

$P_n = \pi_{n+1} + vp_{n+1}\pi_{n+2} + v^2p_{n+1}p_{n+2}\pi_{n+3} + \ldots$ = value of the future premiums at the expiration of n years.

$R_n = B_n - P_n$ = Reserve at the expiration of n years.

Then, at the commencement of the nth year, the present value of the interest profit receivable at the end of the year will be

$$v_n(i_n - i)(R_{n-1} + \pi_n) = v_n(i_n - i)(vp_nR_n + vq_nS_n)$$
$$= vv_n(i_n - i)(p_nR_n + q_nS_n) = (v - v_n)(p_nR_n + q_nS_n) \quad . \quad (8)$$

Similarly, at the commencement of the nth year, the present value of the mortality profit for that year will be

$$v_n(q_n - q'_n)S_n - v_n(p'_n - p_n)R_n \quad . \quad . \quad . \quad . \quad (9)$$

Adding expressions (8) and (9), we shall obtain

$$S_n(vq_n - v_nq'_n) + R_n(vp_n - v_np'_n) \quad . \quad . \quad . \quad (10)$$

† In reckoning the years 1, 2, 3 ... the starting-point is the date on which the present value is to be calculated, which need not be the date of the commencement of the assurance.

which must be multiplied by $v_1v_2 \ldots v_{n-1}p'_1p'_2 \ldots p'_{n-1}$ to obtain the present value at the beginning of the *first year.* Doing this and summing from $n=1$ upwards, we have, after expanding R_n,

$$S_1(vq_1 - v_1q'_1) + (vp_1 - v_1p'_1)[(vq_2S_2 + v^2p_2q_3S_3 + \ldots) - (\pi_2 + vp_2\pi_3 + \ldots)]$$

$$+ v'_1p'_1S_2(vq_2 - v_2q'_2) + v'_1p'_1(vp_2 - v'_2p'_2)[(vq_3S_3 + v^2p_3q_4S_4 + \ldots) - (\pi_3 + vp_3\pi_4 + \ldots)]$$

. . . .

. . . .

The coefficient of S_n in the total will be

$$q_n[(vp_1 - v_1p'_1)v^{n-1}p_2 \ldots p_{n-1} + v_1p'_1(vp_2 - v_2p'_2)v^{n-2}p_3 \ldots p_{n-1} + \ldots] + v_1v_2 \ldots v_{n-1}p'_1p'_2 \ldots p'_{n-1}(vq_n - v_nq'_n)$$

which cancels out and reduces to

$$v^np_1p_2 \ldots p_{n-1}q_n - v_1v_2 \ldots v_np'_1p'_2 \ldots p'_{n-1}q'_n.$$

Similarly, the coefficient of π_n will be

$$-[(vp_1 - v_1p'_1)v^{n-2}p_2p_3 \ldots p_{n-1} + v_1p'_1(vp_2 - v_2p'_2)v^{n-3}p_3 \ldots p_{n-1} + \ldots + v_1v_2 \ldots v_{n-2}p'_1p'_2 \ldots p'_{n-2}(vp_{n-1} - v_{n-1}p_{n-1})]$$

$$= -(v^{n-1}p_1p_2 \ldots p_{n-1} - v_1v_2 \ldots v_{n-1}p'_1p'_2 \ldots p'_{n-1}).$$

Thus the whole expression reduces to

$$[S_1(vq_1 - v_1q'_1) + S_2(v^2p_1q_2 - v_1v_2p'_1q'_2) + \ldots] - [\pi_1(1-1) + \pi_2(vp_1 - v_1p'_1) + \ldots]$$

$$= R_0 - R'_0. \qquad (11)$$

where R'_0 indicates that the reserve is calculated at the experience rates of mortality and interest, the premiums valued remaining unchanged. If, then, R''_0 represent the reserve on the experience basis, *assuming the premiums payable to be different from those employed in obtaining* R_0, we shall have the most general expression

Present value of profit from mortality and interest

$$=R_0-R''_0+\text{[present value, at experience rates, of the difference between the premiums used in the valuation and the corresponding premiums used in determining } R''_0] \quad . \quad (12)$$

From this general expression we may deduce the results for any particular class of benefit. Thus, putting $S_1=S_2=S_3\ldots$, $\pi_1=\pi_2=\pi_3\ldots p_1+q_1=p'_1+q'_1=p_n+q_n\ldots=1$, we have the case of an ordinary whole-life assurance; again, putting $\pi_1=\pi_2\ldots=0$, we have the case of a paid-up whole-life assurance, and the formula reduces to $A_0-A'_0$. For annuities, we must put $q_0=p_0$, $q'_0=p'_0\ldots$, and the expression becomes $a_0-a'_0$, and so on.

Discussion.

The PRESIDENT (Mr. A. J. Finlaison, C.B.), said that Mr. Lidstone had given fresh interest to a subject which had been discussed through many volumes of the *Journal*. The late Mr. Jellicoe (*J.I.A.*, I, 22*, 159), had emphatically indicated his opinion of the proper principles which should regulate the division of surplus; but the general public who, after all, finally controlled the matter, did not apparently want the distribution of surplus to be made according to the principles of association which underlay the theory of life assurance. They wanted increasing reversionary bonuses, or their equivalents, and appeared to desire that that part of their premiums not actually required to produce the bare sum assured should, to a certain extent, be applied as a solatium to those who live longer than the average time.

Mr. GEORGE KING said the business of life assurance was in a transition state at the present time. A serious fall had taken place in the rate of interest, which fall had been gradual, but he was not sure that it had not been accelerated recently. They had to face that difficulty in connection with the management of life offices. Also, of late years the rate of mortality had not been so stable as at one time they had been accustomed to view it. They had had the influenza epidemics which had affected some companies to a very sensible extent, and had made a most unexpected bend in the curve of the rate of mortality. Then a number of old companies had been awakening, and had been seeking new business, and obtaining it. The result was that the conditions of life offices had very much altered of recent years, and the altered conditions necessitated altered views of assurance finance. Still another difficulty was the increased cost of life assurance business. To what that increased cost was due he would not at present refer,

but he thought they all felt it, and if they would not incur it the business disappeared. All those things told, not on the solvency of the companies, because the solvency of British companies was beyond suspicion. but on their bonuses: therefore any investigation into the question of bonuses at the present time was specially opportune. He agreed with Mr. Lidstone, and he thought he might say with Dr. Sprague himself, that the modification of his (Dr. Sprague's) method by adopting the loading instead of the premium for allocating the general surplus was a great improvement; it made the method fit in better to varying circumstances and to be more adapted to general use. It was also the fairer as between the various ages at entry, as the author had brought out. It also suited better the distribution of profits as between the whole-life cases and endowment assurances. In the matter of the interest profit, endowment assurances took, he thought fairly, the lion's share, but if they divided the general profits as well in proportion to the premiums. endowment assurances certainly took too much, because they received part of the general profits in respect of the premiums they paid purely for the sake of the endowment. Another point of some importance was the change in the tables of premiums. Some companies might have adhered to the same rates of premiums throughout the whole of their existence, but not many had done so. From time to time it had been found convenient to change the rates of profit premiums, and if they went on the premium basis that change was not given sufficient effect to; whereas if they went on the loading basis, each of the various sets of premiums received its fair share of surplus. The next point was that of the data which should be employed in ascertaining the loading. He agreed with the author that they must discard the valuation net premium unless it was one of a special and suitable kind, and adopt a net premium which was more equitable, and he thought that, for that purpose, the select net premium was the best, at the rate of interest, however, used in the valuation. It was quite true that for valuation purposes they had practically the same result, whether they adopted the select net premium or the H^M net premium, valuing by the H^M and $H^{M(5)}$ Tables, but that was taking the broad average. When they came to distribute the divisible surplus among individual policies they ceased to have averages, and a table which merely gave a fair average was not an equitable one for such a purpose. It seemed to him that the loading method was more appropriate in connection with endowment assurances, and he was glad to find that the result of Mr. Lidstone's investigations was to show that endowment assurances were really entitled to practically the same reversionary bonuses as whole-life policies. He knew that was a controversial question, but it was a very important one in view of the great increase in recent times of endowment assurances. His own view for some time had been that they were entitled to equal reversionary bonuses, of course assuming that the rates of premiums were suitable. And he found that the competition rates for whole-life cases and for endowment assurances really were such as would give to endowment assurances practically the same, or perhaps a little larger, reversionary

bonus as to whole-life policies. Those remarks applied to endowment assurances with even as limited a period as ten years. That result had been arrived at by a totally different method of investigation from that which Mr. Lidstone had adopted. By discounting a certain amount of bonus on each policy he (Mr. King) found there was a larger margin of loading left on endowment assurances than in whole-life cases. Certain companies also which used Dr. Sprague's method and allotted the bonuses strictly according to that method among the different classes of policies, gave a large bonus to endowment assurances. He could also point to the same result being produced in a totally different way; in the case of one or two companies which kept the two sets of policies in entirely distinct accounts, treating each set as if it were a separate company, the result was that the endowment assurances received very large bonuses. Surely, when the same conclusion was reached by four different methods of investigation there was some reason to think that they had got at the truth, and that endowment assurances should receive practically the same reversionary bonuses as whole-life cases. Dr. Sprague's method was not a method that as a rule he would like to advise for adoption, and there were very strong reasons that he could adduce for that position. The question was, was the method elastic? Some actuaries said that it was; and he thought for fair weather sailing it was an extremely good method; but it had the effect of throwing all the fluctuations upon the younger policies—fluctuations in mortality, fluctuations in general profit, all fell upon them. The change in the rate of interest was so gradual that they could allow for it, and they had practically no fluctuations in respect of the profits from surplus interest. He could conceive of a company which should show a very heavy rate of mortality, say, on account of influenza, and that that heavy rate should fall upon the old lives, while the young lives showed a particularly favourable rate of mortality, and yet, on account of the losses from mortality, or absence of gain from mortality, being thrown upon the general profits—and the young lives taking the bonuses almost entirely from the general profits—the company would give very little bonus on the new policies. He thought that a very great objection. Then, how were they to deal with the strain upon the company if it were found desirable to reduce the rate of interest in the valuation. Practically, by Dr. Sprague's method of distribution, the whole of the reserves set aside for increasing the stringency of the valuation, must come out of the profits of the recent policyholders, and yet it was not for their sake alone that that increased stringency had been imposed, but for the general good of the company. There was a want of elasticity in that method of division which was a little awkward. Another point of rather serious consequence was the reserves made for new business. He maintained that the reserves of a company should not be set against individual policies, but that they were for the general good, and that it did not matter much how they calculated those reserves so long as they had them. By the usual method of calculating reserves, new policies did not show much profit, although there was not a loss upon the business—he was not speaking of expenses, because the expenses to a considerable extent might be set off against

the medical selection, he was speaking of the reserves which were made for new business. New business produced but little surplus at the beginning, and therefore it was a considerable difficulty to make surplus, especially with methods such as that of Dr. Sprague. From these considerations he came to the conclusion that unless the company was a large one carrying small limits, with risks widely distributed, and doing a very uniform amount of business, the method was not very desirable. It was apt to produce awkward breaks of continuity in the bonuses, and to cause fluctuations to fall upon that portion of the business which managers for the most part wished to favour, and therefore he did not think it was a method desirable in the majority of cases in practice. It was an extremely useful method to measure other methods of distribution by, but it was not one that he would advise for adoption except under special circumstances. He would now bring forward, as briefly as possible, an investigation which he had recently made into the question of the profits on new business, and when those profits began to be realized. Let Company A value at rate of interest i, and let the net premium on that basis be P_1, and let the policy-value be represented by V. Let Company B value by a higher rate j, with corresponding net premium P_2, and policy-value U. We then have the self-evident equations

$$\text{A.}\quad (V+P_1)(1+i)=q+pV_{+1}$$

$$\text{B.}\quad (U+P_2)(1+j)=q+pU_{+1}$$

which simply tell us that the reserve at the beginning of the year, with the net premium then paid, accumulated at the valuation rate of interest, is equal to the claims of the year and the reserve at the end of the year. Now, assume for the moment that both companies realize rate j. Then Company A by making the higher reserves loses in the year, as compared with B, loading to the amount of $(P_1-P_2)(1+j)$, and gains in interest $(V+P_1)(j-i)$. In the early days of a policy the loss will be greater than the gain, and A will have the smaller divisible surplus, while later on the position will be reversed. The two companies will be on a par in this respect when $(P_1-P_2)(1+j)=(V+P_1)(j-i)$, that is, when $V(j-i)=P_1(1+i)-P_2(1+j)$, that is, when

$$V=\frac{P_1(1+i)-P_2(1+j)}{j-i}.$$

By calculating out the expression for any age at entry, and referring to a table of policy-values, the time of equal surplus can at once be found. Thus, taking age 35 and rates of interest 3 per-cent and 4 per-cent, and using the H^M Table, we find that, at the moment of equality of surplus, $V=\cdot 211$, corresponding to a duration of between 12 and 13 years of the policy. For 12 years, Company A, making the more stringent valuation, will have the smaller surplus, but after 13 years it will have the greater. For younger ages at entry the time will be longer, and for older ages shorter. Taking now the more general case of both companies valuing below the rate of interest realized, which may be written J, so that $J>j>i$, with

Company A the loss of loading will be $(P_1-P_2)(1+J)$, and the gain from interest will be $(V+P_1)(J-i)$, so that the net gain or loss, as the case may be, will be

$$\begin{aligned}&(V+P_1)(J-i)-(P_1-P_2)(1+J)\\ &=V(J-i)-P_1(1+i)-P_2(1+J).\end{aligned}$$

With Company B there will be no loss of loading, and the gain from interest will be $(U+P_2)(J-j)$. The companies will be on an equality when these expressions are equal, that is, when

$$V(J-i)-P_1(1+i)-P_2(1+J)=U(J-j)+P_2(J-j),$$

that is, when

$$V(J-i)-U(J-j)=P_1(1+i)-P_2(1+j).$$

In this more complicated case the time cannot be found by simple reference to a table of policy-values, but by making trials it can be ascertained by interpolation. Thus, taking age 35 as before, and rates of interest 4 per-cent, 3 per-cent, and $2\frac{1}{2}$ per-cent, and assuming first 10 years, we have

$$\begin{aligned}V(J-i)-U(J-j)&=\cdot 00093,\\ P_1(1+i)-P_2(1+j)&=\cdot 00121,\end{aligned}$$

so that at the end of 10 years Company A still yields less surplus. At the end of 15 years $V(J-i)-U(J-j)=\cdot 00142$, so that now A yields the larger surplus, and by interpolation we find that the point of equality is reached between 12 and 13 years.

Mr. G. F. HARDY said that Mr. Lidstone's numerical results depended upon the particular basis of premiums which he had adopted, but they were average premiums, and probably his results would not be greatly modified in the case of any of the better known offices. At all events, the author had given them the means of making those modifications if they desired, and they could readily pass from the average premiums which he had adopted as the basis of his paper to any particular scale of premiums which they might wish to discuss. The paper raised the whole question as to how far surplus could be equitably returned to the assured. It had always been a more or less burning question, and a very great difference of opinion had from time to time been expressed thereon. What standard were they to aim at? He thought that the remarks of Mr. Browne, referred to on page 89, gave them a very good test as to the sort of standard which they should endeavour to aim at in practice, that was, that the method of distribution should be of such a character that any change or fluctuation in the nature of their business, either as regarded the average age of entry or table, should leave them in the position of being able to declare an undiminished bonus. It was not enough that their rates should on the average be sufficient to meet the bonuses which they were declaring; they must have rates which were practically sufficient in every individual case because the assuring public was reducing insurance by the aid

of agents, brokers and so forth to more or less of a fine art, and there were very few policies now effected in which there was not a preliminary canvassing of rates and other advantages given by different offices. Therefore, if any given office had a table in which the rates were specially low and the bonuses were specially advantageous, there was almost certain to be an excess of business done under that table, and the average bonuses declared would suffer in consequence. When they came to the individual sources of profit they saw that the interest profits were most simply determined and most simply divided. Interest profits, as Dr. Sprague had shown, should be returned in proportion to the sums held on the average against the individual policies. In arriving at those interest profits, however, a point had to be considered—namely, that they should be debited with a portion of the expenses. If they had a large fund to invest, it would be impossible to conduct the business without a considerable expenditure, and that portion of the expenses of management which was due to the investing of the funds must be in many offices considerable, and should be taken into account in deciding what were the profits from interest. An office, for example, which was making $4\frac{1}{4}$ per-cent on its funds might very reasonably set aside at least $\frac{1}{4}$ per-cent representing the cost of gaining that surplus rate of interest. The crux of the method of distribution which they owed to Dr. Sprague was the question of distribution of the general profits. He was not prepared to agree with Mr. King nor with the author that the substitution of loadings for premiums entirely met the case. The main source of profit after the profits from loadings and interest had been exhausted appeared to be nowadays the profit from light mortality, and no doubt it was difficult to distribute that profit perfectly equitably, especially as between entrants at various ages and under various tables. Taking the various tables, such as the whole-life and endowment tables, the mortality profits were not in proportion to the premiums paid. For example, take the simple case of the whole-life table and the endowment table. In the endowment table the premiums paid were very much larger, but the profits of mortality were unquestionably much smaller, so that as a matter of fact the shorter the endowment the less the profit from mortality until they got to the extreme case of very short endowments, with no time at all for profit from mortality to accumulate. In this very important particular the substitution of loading for premiums was therefore quite inadequate to meet the requirements of the case. Under existing conditions, with a rate of interest still keeping up a little above 4 per-cent, he was prepared to agree that endowment assurances were entitled to the same reversionary bonuses as whole-life policies; but if the rate of interest should fall in the future, the main source from which profits on endowment assurances were drawn would disappear; whereas in the case of the whole-life policies there was a source of profit from the probable light mortality which would very likely quite compensate for the fall in the rate of interest. The distribution of profit arising from non-profit business would depend very largely on the general resources of the company. The profit drawn from this source should equitably go in much the same direction with the interest profit, and this might perhaps

justify the setting of such profit against certain expenses of management which might fairly be attributed to the interest income. The series of tables given were particularly interesting, as showing clearly that after all the compound reversionary bonus system nearly met the equity of the case. He was pleased to find Mr. Lidstone had arrived at that result, because he had himself considered the point, and had arrived at the same conclusion, that with average premiums the compound bonus system did substantial justice not only as between policyholders of different durations and ages, but also as between policyholders of different classes of assurance. Both the appendices were extremely interesting, and the algebraical work very elegant. He was not aware that any attempt had been previously made to give expression to the very interesting quantity the present value of the future profits, and the algebraical demonstration would be very useful. He was inclined to agree with the author, that the final result was more or less self-evident. Taking the case of one company earning a rate of interest i' and valuing at that rate, and another company earning the same rate of interest and valuing at the rate i, the difference between the sums they held in hand at any time as to any policy must be the expression Mr. Lidstone had put down, $R_0 - R''_0$. The difference between the sums the two offices would hold, must evidently be equivalent to the bonus which would be distributed in future to the policyholder of the office valuing at the lower rate of interest. Putting it in that way might bring out a little more strongly the verbal meaning of the result the author arrived at algebraically. He hoped that at some future time Mr. Lidstone would give an equally thorough investigation of the subsequent points to which he had only at present referred.

Mr. A. H. Bailey referred to the early days of the Equitable when it was found that charging 15 per-cent on Northampton premium rates yielded a considerable surplus, which was distributed without much thought as to the equitable division; the plan adopted being to increase the sum assured by a uniform rate per-cent according to the years the policy had been in force. The general public found that there was such a large amount of surplus unappropriated that there was a great rush for policies to share in a fund to which they had not contributed. Then came the celebrated bye-law in 1815. It was then determined that only the 5,000 senior members should participate, and not only so, but that a man should rank not at the date of his policy, but the date of his becoming one of the first 5,000, which in those days required about 15 years. The injustice of this was so obvious that the matter was a good deal discussed and a system was adopted which in his early days was regarded as the only scientific system. It was thus described in a Parliamentary Return: "The difference between the accumulated premium on an assurance supposed to commence at the date of the last valuation and at the age next birthday on that date, and the sum to be reserved as the present value of the liability under such proposed assurance. If the assurance commences since the last valuation the estimate is made for the time the assurance has been in force." There were then none of these varieties of rates of interest and tables of mortality; there was only one table of mortality to be adopted and

one rate of interest throughout. In all actuarial matters in practice they had to discriminate between what was theoretically correct and what was practically expedient, and in this particular subject it was necessary to devise a method which should be intelligible to the public, who did not know what "loading" meant and never would; and there was little doubt that a uniform reversionary bonus was the one that was the most popular. The best course to pursue, he thought, was to distribute the bonus according to the premiums paid during the quinquennium. That did as nearly substantial justice as seemed practicable, and therefore it seemed to be the best method that could be adopted.

Mr. T. G. C. Browne wished to emphasize the remarks made by Mr. King with regard to the method of division associated with Dr. Sprague's name, in saying that it was entirely a fair-weather method of division. It was obvious that if there was a heavy loss from mortality or investments, such loss under that system must fall upon what was termed the general profit, and the result upon the bonuses of the more recent entrants would be simply disastrous. He (Mr. Browne) had advocated a system which created a third group to be divided ratably among the other two. In the first group they had loadings to which surrenders and lapses might be fairly added, though he demurred to a detail in Mr. Lidstone's paper where he charged surrenders and lapses with the expenses. In these days of non-forfeiture schemes their profits from lapses at any rate might become a vanishing quantity, but their expenses unfortunately did not tend that way. He would therefore say that the loadings should be primarily charged with expenses and commission, and any profits from surrenders or lapses should be simply treated as a "grant in aid" of expenses. In the second group there was the surplus interest, and though Mr. Hardy suggested that there should be some charge for expenses, that would not amount to very much. Then came the question, what were they to put in the third group? Practically, the whole of the remainder of the profit—mortality; interest on accrued surplus (a very important item, where they divided every five years); profit from reversionary bonuses on re-assurances; and profit from investments, including profit from reversions above an assumed rate of interest. In the old days, it was customary to under-value the stocks included in reversions purchased very considerably, and if those stocks were good the recent rise in their value brought in an unexpected profit amounting to a very substantial sum. He had had to apply the system he had described at four quinquennial divisions under very varying conditions as regards mortality, where Dr. Sprague's method would certainly have broken down. Whether the system would bear the strain of a continuous fall in the rate of interest was a problem of the future, but he thought it would, as it possessed a considerable amount of elasticity.

Mr. J. Chisholm said, however praiseworthy the desire of actuaries might be to deal out what was called justice amongst the different classes of policyholders, the concentration of the mind on the division of surplus had the evil effect of taking their attention away from other very important considerations. If what the public

wanted was an increasing reversionary bonus, it was a very much simpler plan to give it to them and then to charge a premium that would produce it A merit of the paper before them and of previous papers on the same subject was that they all converged in showing that average premiums produced results in the way of bonus very similar to those that came from the application of the compound reversionary system. One way in which the adoption of fanciful systems of bonus division seemed to have worked evil was that not sufficient regard was given to the premiums paid by the assured. So much surplus had arisen, and an attempt was made to divide it equitably among the various classes of policyholders, but individuals might come off very badly or the reverse. There were rates of premium in vogue among many offices that certainly were hardly to be justified by recent investigations into select mortality; and it was no secret that assurances might be obtained in some cases at less than the cost price. He thought this arose from the fact that an undue amount of attention had been given to bonus distribution, and it was a pity that so much time should have been abstracted from the more important question, what were the proper premiums to charge.

Mr. MANLY referred to Mr. Hardy's statement, that taking the average scale of premiums he had found that the compound reversionary bonus gave an equitable system of distribution. and said that he (Mr. Manly) ventured to think that that was conditional on there being a certain difference between the rate of interest realised and that used in the valuation. He was very pleased to see that Mr. Lidstone had satisfactorily proved his method by the inverse process of finding what the premiums should be, assuming that the benefits he had produced were determined beforehand, and using the actual rate of interest earned.

The PRESIDENT having proposed a vote of thanks to the author, which was unanimously accorded,

Mr. LIDSTONE, in reply, acknowledged the vote of thanks, and said he was glad Mr. King agreed with the substitution of loading for premiums, and also that he was inclined to support the *select* loading system. Mr. King had referred to what he might call the "cast iron" nature of Dr. Sprague's method, and he had said that it was extremely inconvenient if there were considerable fluctuations of mortality. He (Mr. Lidstone) thought that Mr. King was speaking rather from the managerial standpoint, having regard to the effect it would have upon new business. He did not know that he could altogether follow Mr. King's argument, for it seemed right that the younger lives should bear fluctuations to a large extent, because they had the opportunity of surviving through more prosperous periods, and thus having compensation which the older lives would not have. His investigation showed that under certain conditions the compound reversionary bonus system was equitable; but he quite agreed with Mr. Manly that there must be a margin of about 1 per-cent on the rate of interest to keep it up. It was not only a question of what the premiums would earn but how they would earn it, *i.e.*, what the *incidence* of the bonuses would be. Mr. Browne's method was an extremely interesting one. He had looked into it and had intended

to refer to it in his paper, but he had found that it would require a great deal of investigation which he was unable at the time to undertake. He was not quite prepared to say how far it was theoretically defensible, although he quite admitted that it might be practically more convenient than Sprague's method if the mortality fluctuated violently.

CORRESPONDENCE.

MAXIMUM MORTALITY PERCENTAGES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—I trust that I will not be considered presumptuous if I venture to offer a few words of friendly comment on a portion of a paper which has deservedly taken its place as one of the classics of actuarial science, and which has as its author one of the most honoured leaders of the profession. I refer to the essay by Dr. T. B. Sprague, "On the rate of mortality prevailing among assured lives, as influenced by the length of time for which they have been assured" (*J.I.A.*, xv, 328).

In that paper Dr. Sprague pointed out that in the experience of the Twenty Offices (H^M) a maximum mortality percentage is apparently reached between the eleventh and fifteenth years of assurance, and that this supposed fact may be accounted for by the operation of two factors, (1) the gradual wearing out of the beneficial effect of the medical examination at entry, and (2) the effect produced by the withdrawal of healthy lives What I wish to suggest now is, whether the peculiarity in question is not capable of an entirely different explanation, and due to an entirely different cause.

In an investigation into the effects of selection, the plan of comparing the actual number of deaths with those predicted by a standard table is certainly an excellent one. It is, however, essential that the table selected as a standard be one which is in every way suitable and reliable. If the gauge by which we measure be one which is not applicable, the results arrived at by its use will, in all probability, be incorrect. Dr. Sprague himself recognized this when he said, "It occurred to me that the fact of grouping together persons of all ages at entry might have a disturbing effect on the results, as would clearly be the case if the table of mortality from which the probable deaths are computed gave much too high a mortality at the early or middle ages." A careful examination will, I think, convince us that this is exactly what has happened, although the predicted mortality is too low, instead of too high, at the early and middle ages. Any table deduced in the ordinary way from the experience of life companies is itself influenced by selection, and is therefore from its very nature an improper standard. During early and middle adult life, an addition is made at each age of a

Percentage of Actual Deaths to those computed by the H^{M} Table.

Years of Assurance	Present Ages																All Ages
	15–20	21–25	26–30	31–35	36–40	41–45	46–50	51–55	56–60	61–65	66–70	71–75	76–80	81–85	86–90	91–96	
0	52·9	40·6	41·4	63·5	40·4	37·5	44·9	44·0	46·6	25·1	49·9	16·1	...	...	...	...	44·6
1	...	88·2	79·0	72·3	63·1	70·9	68·8	51·4	...	62·4	63·3	79·9	...	...	...	...	71·5
2	118·7	...	104·2	91·5	85·3	91·0	86·3	83·6	79·4	73·2	69·8	...	...	...	...	...	88·9
3	...	105·9	109·6	105·2	100·2	96·0	99·4	93·4	...	...	...	89·9	...	...	...	...	98·6
4	...	138·8	133·7	...	...	...	100·1	...	88·4	...	...	...	51·5	...	...	...	108·4
5	141·9	162·8	...	108·8	106·5	100·8	101·1	96·7	94·8	99·2	92·3	99·8	85·4	...	...	...	101·3
6–10	...	186·3	135·6	116·6	114·1	106·2	104·9	99·8	101·4	101·7	104·6	101·3	92·7	...	...	...	105·1
11–15	...	...	140·3	126·7	120·2	107·5	108·8	108·9	108·7	104·6	105·2	105·1	96·7	93·3	...	...	108·7
16–20	...	...	...	...	118·9	124·9	109·2	109·7	108·2	106·4	105·8	105·2	100·1	93·5	89·8	...	105·2
21–25	...	...	...	...	...	131·4	109·7	113·6	102·3	103·3	101·1	...	101·3	95·8	...	...	104·5
26–30	...	...	...	...	...	...	106·8	107·7	...	103·1	...	102·5	104·4	105·9	97·6	...	103·5
31–63	...	...	...	...	...	...	...	...	93·6	100·0	99·7	90·7	103·1	109·3	107·9	108·2	102·1
TOTAL	104·6	95·5	103·0	98·6	100·9	99·7	100·3	99·8	99·5	100·2	100·0	100·8	99·5	101·2	99·1	108·2	100·07

large number of newly-selected risks. At ages 20 or 25 the lives exposed are all practically fresh from the examiners' hands.

As age advances the proportion of these new cases to the total becomes steadily less, and after 65 the lives exposed may be considered as, for practical purposes, containing none which have been recently admitted. The consequence is that the mortality shown by the table during the earlier years of life is unduly depressed by this artificial condition, and that the table as a standard becomes increasingly severe with advancing age. It may seem out of place to refer to such a well-known fact, but I do so because I think it is the key to the whole problem.

Let us now examine the percentages brought out by Dr. Sprague. I regret that it is impossible to arrange them under uniform groupings, but I have endeavoured to do this as nearly as possible. In each set I give in their exact order all the percentages. Those who wish to examine the slight variations which have been made in the groupings are referred to the original paper.

[*See preceding page.*]

In the above table the maximum in each set is printed in heavier type, while the percentages for the years 11 to 15 are enclosed within parallel rules to assist the eye. If we glance from left to right along the line opposite to any year of assurance we will be struck by the marked and steady reduction which takes place in the percentages. Taking, for example, the years 11 to 15, we find a gradual fall from 140·3 for ages 26 to 30, to 108·9 for ages 51 to 55, and to 93·3 for ages 81 to 85. This is, of course, what we would expect. It does not necessarily mean that the mortality at the higher ages is more favourable than at the younger, but rather that the mortality table by which the expected deaths are computed provides for a greater number of deaths proportionately at the older ages than at the younger.

Possibly, however, it may appear to some that this statement is inconsistent with the fact that the percentages for all durations combined as given at the foot of each column are all very close to 100, and are about equally heavy at all ages. It is merely necessary to point out that this is because the majority of deaths at the younger ages occur under policies of short duration, while at the higher ages the majority are under policies of long duration.

We now note several points brought out by the above table:

(1) Discarding ages over 90, seven of the 15 sets show an uninterrupted progression from lower to higher percentages, the highest percentage of all being in each case the last one in the group.

(2) Four of the remaining sets show an uninterrupted progression until the last percentage but one. They show a falling off in only the last item. The numbers of deaths on which the last percentages in these four sets are based are respectively 36, 23, 221, and 119, making in all only 399 deaths out of a total of 7,344 at those ages. Whether a uniform grouping of the durations or a very

slight re-arrangement of them might not cause the supposed maxima to disappear entirely from these groups is a matter about which no definite opinion can at present be expressed. At any rate they are evidently of but slight importance.

(3) Of the remaining four sets one only shows a maximum falling between the eleventh and fifteenth years of assurance, the other three falling later.

It is therefore clear apparently that so far as the subdivided ages are concerned, they do not, on the whole, confirm the maximum mortality theory, but on the contrary, tend to show that no such maximum exists—certainly not between durations 11 and 15.

How then does the summary for all ages combined show a maximum between years 11 and 15? Simply because by the uniting of all the divided sets the peculiarity of the mortality table comes into play, and the percentages for the longer durations of policies become more favourable, because these are the percentages which belong also to the older ages of life, where they are, as we have already shown, naturally and of necessity lower than at the younger ages.

Perhaps a practical illustration will be the most satisfactory way of explaining the matter. Let us take two hypothetical groups and then combine them.

Hypothetical Illustration.

Years of Assurance	Present Ages 36-40			Present Ages 66-70			Combined Ages		
	Expected Deaths	Actual Deaths	Per-cent	Expected Deaths	Actual Deaths	Per-cent	Expected Deaths	Actual Deaths	Per-cent
1–10	1,000	800	80	100	62	62	1,100	862	78·4
11–15	1,000	1,150	115	500	500	100	1,500	1,650	110·0
16–20	200	250	125	1,000	1,020	102	1,200	1,270	105·8
21–30	...	...	...	600	618	103	600	618	103·0
Total	2,200	2,200	100	2,200	2,200	100	4,400	4,400	100·0

In this illustration we have two separate groups of lives, each having a mortality which agrees exactly with the table. Neither shows any maximum, but on the contrary both get progressively worse. When, however, they are combined they produce percentages which on their face would indicate a maximum, although such does not exist in reality at all.

It may, however, be objected that as this combination is a mere supposition, it may perhaps have no bearing on Dr. Sprague's tables at all. To settle this doubt, I have picked out and combined the seven sets of lives which admittedly show no maximum (ages 15–20, 21–25, 26–30, 31–35, 41–45, 81–85, 86–90). As the groupings are

not identical I have had as before to place all irregular durations into the group to which they most nearly belong. The results are as follows:

Percentage of Actual to Computed Deaths.

Years of Assurance	Expected Deaths	Actual Deaths	Percentages	
0	345·90	164	47·4	47·4
1	603·53	459	76	76
2	488·38	469	96·1	96·1
3	509·63	528	103·5	103·5
4– 5	1,156·16	1,247	107·9	107·9
6–10	920·44	1,053	114·5	114·5
11–15	364·72	399	109·4	110·7
16–20	232·49	262	112·7	
21–25	127·01	139	109·4	109·4
26–30	117·34	121	103·2	106·5
31–63	175·27	191	108·9	
TOTAL	5,040·87	5,032	99·8	99·8

We here see in actual operation the principle referred to. Not one of these seven groups shows any maximum when considered by itself, and yet, when they are combined, there is a very marked maximum observable between the sixth and tenth years. This is conclusive proof that so far at least as this section of the experience is concerned, the supposed maximum is due solely to the nature of the table used as a standard. The addition of one other set (36–40) would have been sufficient to bring the maximum forward to the 11th–15th years of duration.

If it be supposed that as the Peerage Table is not founded on assurance experience it therefore is a proper standard for use in such a case as the present, and that the objections which we have been urging cannot apply to it, I would point out (1) that the Peerage Table is based solely on one particular stratum of society and is therefore inapplicable, and (2) that its mortality curve follows a peculiar course, being high at ages under 30 and over 70, and low between these ages. It would naturally therefore give results very similar to those of assurance tables, and like them must be discarded.

From a consideration of these facts, I think we must come to the conclusion that, much as we must all respect Dr. Sprague's opinion on this or any other point, the statistics which we have been examining contain no satisfactory evidence of the existence of such a maximum mortality epoch as has been supposed.

Yours truly,

T. B. MACAULAY.

Montreal,
10 *April* 1895.

THE LIFE ASSURANCE COMPANIES OF THE UNITED KINGDOM.

Summary of the Life Assurance and Annuity Revenue Accounts.

[Extracted from the Parliamentary Return for 1894, published in 1895.]

INCOME	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Balance at the beginning of the Year	181,692,907	10,816,075	192,508,982
Premiums	16,862,514	5,927,835	22,790,349
Consideration for Annuities . .	1,415.769	791	1,416,560
Interest and Dividends (less Tax)	7,252.747	368,639	7,621,386
Increase in value of Investments .	189,614	3,739	193,353
Fines, Fees, &c.	10,383	869	11,252
Capital Paid-up	16,361	101,406	117,767
Customs Timber Measuring, &c. .	2,667	...	2,667
Donations (Itinerant Methodists) .	1,125	...	1,125
Transfers from other Accounts .	59,462	30,732	90,194
Miscellaneous	157	450	607
	207,503,706	17,250,536	224,754,242

OUTGO	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Claims	12,774,929	2,547,832	15,322,761
Cash Bonuses and Reduction of Premiums	1,085,490	27	1,085,517
Surrenders	1,006,851	20,393	1,027,244
Annuities	1,054,276	2,418	1,056,694
Commission	916,306	1,577,803	2,494,109
Expenses of Management . .	1,588,123	994,927	2,583,050
Bad Debts	4,168	155	4,323
Decrease in value of Investments .	35,120	209	35,329
Interest on Capital and Dividends and Bonuses to Shareholders .	531,004	231,256	762,260
Transfers to other Accounts . .	76,917	93	77,010
Capital withdrawn from Life Account (Pioneer, Limited) . .	27,500	...	27,500
Miscellaneous	30,486	...	30,486
Balance* at the end of the Year .	188,372,536	11,875,423	200,247,959
	207,503,706	17,250,536	224,754,242

* This Balance includes the whole of the Life and Annuity Funds (£194,175,429), and, in addition, the Capital of Companies whose business is limited to Life Assurance only.

Summary of the Balance Sheets (1894).

LIABILITIES	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Paid-up Capital (including sundry Shareholders' Balances)	11,197,946	963,503	12,161,449
Life and Annuity Funds	183,009,704	11,165,725	194,175,429
Fire Funds of Companies transacting Life Business	9,684,620	.	9,684,620
Marine Funds of Companies transacting Life Business	618,490	...	618,490
Reserve Funds	4,129,264	...	4,129,264
Other Funds	791,519	250,314	1,041,833
Profit and Loss Balances	2,573,335	...	2,573,335
Depreciation and Investment Balances	1,022,698	...	1,022,698
Globe Annuitants (Liverpool and London)	1,102,800	...	1,102,800
Outstanding Claims	4,167,966	40,074	4,208,040
Outstanding Accounts	419,876	19,669	439,545
Temporary Loans	204,935	...	204,935
	218,923,153	12,439,285	231,362,438

ASSETS	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Mortgages	84,743,199	399,421	85,142,620
Loans on Policies	10,270,509	29,217	10,299,726
,, Rates	21,420,034	4,783,448	26,203,482
British Government Securities	4,868,072	634,059	5,502,131
Indian and Colonial Government Securities	15,654,471	163,585	15,818,056
Foreign Government Securities	4,162,339	...	4,162,339
Debentures	27,012,987	1,757,038	28,770,025
Shares and Stocks	14,415,667	19,212	14,434,879
Companies' own Shares	606,415	...	606,415
Land and House Property and Ground Rents	14,781,487	3,527,839	18,309,326
Life Interests and Reversions	4,150,874	427	4,151,301
Loans on Personal Security	1,551,254	8,277	1,559,531
Agents' Balances and Outstanding Premiums	5,070,923	423,931	5,494,854
Outstanding Interest	2,164,829	130,126	2,294,955
Cash, Deposits, Stamps, &c.	7,853,284	173,697	8,026,981
Customs Timber Measuring Balances, &c.	2,028	...	2,028
Book-Room Grant (Itinerant Methodists)	40,000	...	40,000
Deficiencies, Preliminary Expenses, &c.	154,781	389,008	543,789
	218,923,153	12,439,285	231,362,438

Increase (+) *or* Decrease (−) *in the Chief Items of this Year's* Summary *compared with the corresponding Items for the previous Year.*

	Ordinary Companies	Industrial Companies
	£	£
Income.		
Premiums	+ 288,828	+ 218,144
Consideration for Annuities	+ 56,293	− 3,016
Interest and Dividends (less Tax)	+ 45,919	+ 33,350
Net Result of Realization and Re-valuation of Investments	+ 102,069	+ 3,998
Outgo.		
Claims	− 742,139	+ 95,867
Annuities	− 1,135	− 16
Surrenders	+ 68,880	+ 4,484
Commission	− 3,846	− 3,606
Expenses of Management	+ 1,494	+ 5,787
Liabilities.		
Paid-up Capital (including sundry Shareholders' Balances)	− 57,049	+ 103,975
Life and Annuity Funds	+ 6,637,518	+ 959,104
Assets.		
Mortgages (including Loans on Rates)	+ 1,312,381	+ 561,989
Life Interests and Reversions	+ 79,524	− 95
Loans on Policies	+ 516,309	+ 1,591
British Government Securities	− 83,103	+ 131,618
Indian and Colonial Government Securities	+ 1,388,525	+ 16,285
Foreign Government Securities	+ 303,232	...
Debentures	+ 2,246,571	+ 16,913
Shares and Stocks	+ 815,211	− 13,530
Companies' own Shares	+ 2,924	− 442
Land and House Property and Ground Rents	+ 803,526	+ 273,826
Loans on Personal Security	+ 53,336	− 421

NUMBER OF COMPANIES.

The total number of Companies appearing in the above Summary is 95, of which 84 have been classed as Ordinary, 7 as Industrial, and 4 appear in both Classes, the Returns of these Companies showing the Ordinary and Industrial business separately.

During the year two names have been removed from the official List of Companies, namely, the England Assurance Institution; and the Midland Counties Insurance Company; which have transferred their business; and one name has been added, namely, the Absolute Life Assurance Company, Limited; in which case the Board of Trade have issued their Warrant under the provisions of Section 1 of "The Life Assurance Companies Act, 1872."

SUMMARY OF THE ASSURANCES IN FORCE, *as shown by the last Returns of the Compan*

ORDINARY BUSINESS.

	With Profits		Without Profits		Total		Re-assurances	Net Amoun
	No.	Amount	No.	Amount	No.	Amount	Amount	
ASSURANCES.		£		£		£	£	£
Whole Term of Life	722,609	352,150,910	122,957	61,839,843	845,566	413,990,783	20,441,481	393,549,
Limited number of Premiums . .	29,020	17,383,206	5,349	2,576,142	34,369	19,959,348	702,091	19,257,
	751,629	369,534,146	128,306	64,415,985	879,935	433,950,131	21,143,572	412,806.
Endowments . .	1,900	424,964	12,970	2,325,070	14,870	2,750,034	36,109	2,713,
Endowment Assurances . . .	340,000	59,580,714	30,308	9,401,566	370,308	68,982,280	1,026,361	67,955,
Joint Lives . .	12,363	2,529,250	2,174	959,812	14,537	3,489,062	296,436	3,192.
Last Survivor . .	1,110	883,899	1,233	1,333,528	2,343	2,217,427	323,630	1,893,
Contingent . .	35	32,269	3,156	5,327,759	3,191	5,360,028	1,373,504	3,986,
Issue . . .	5	13,249	995	3,879,455	1,000	3,892,704	1,101,780	2,790,
Miscellaneous . .	448	366,597	4,516	5,420,171	4,964	5,786,768	1,120,976	4,665,
	1,107,490	433,365,088	183,658	93,063,346	1,291,148	526,428,434	26,422,368	500,006
ANNUITIES.								
Immediate . .	...	...	...	...	19,991	960,529	10,324	950
Deferred . .	...	..	...	..	6,514	190,076	12,392	177
	...	...	...	...	26,505	1,150,605	22,716	1,127

INDUSTRIAL BUSINESS—(Sickness and Friendly Society Contracts not included).

	With Profits		Without Profits		Total		Re-assurances	Net Amou
	No.	Amount	No.	Amount	No.	Amount	Amount	
ASSURANCES.						£		£
Whole Term of Life	...	...	...	...	12,872,226	121,944,531	...	121,944
Limited number of Premiums . .	...	...	...	...	453	3,528	...	3
	...	...	...	...	12,872,679	121,948,059	...	121,948
Endowments . .	...	...	...	..	93,210	1,254,912	...	1,254
Endowment Assurances . . .	...	...	...	..	156,066	1,635,278	...	1,635
Joint Lives . .	...	...	...	...	202,823	3,225,861	...	3,225
	...	...	...	...	13,324,778	128,064,110	...	128,064
ANNUITIES.								
Immediate . .			..	...	1	15	...	

The above figures are based on Returns deposited, for the most part, during the past five years are, therefore, merely an approximation to the amount of contracts in force at the present time. In case of two Companies, namely, the Co-operative and the Northern, the amount of business at a recent date has been included. The figures of the Colonial and Foreign Companies have been exclude their Returns do not separately show the extent of business in the United Kingdom.

THE INSTITUTE OF ACTUARIES.

EXAMINATIONS OF THE INSTITUTE, APRIL 1895.

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE (PART I).

Examiner—PROF. S. L. LONEY, M.A.

First Paper.

1. A man directs his broker to buy for him nine £100 shares at $253\frac{7}{8}$. To provide a portion of the money required, he directs him to sell £750 Consols at $100\frac{3}{8}$, and £800 of a certain railway stock at $171\frac{5}{8}$. The brokerage being $\frac{1}{8}$th per-cent on each transaction, what sum of money will he owe the broker?

2. Simplify the expression

$$\frac{a^2}{(a+x)(a-b)(a-c)}+\frac{b^2}{(b+x)(b-c)(b-a)}+\frac{c^2}{(c+x)(c-a)(c-b)},$$

and prove that, if n be an odd integer, then $(a-b)^n+(b-c)^n+(c-a)^n$ is always divisible by $(a-b)(b-c)(c-a)$.

3. Solve the equations

$$(1)\quad \sqrt{x^2+ax}+\sqrt{x^2-bx}=\frac{a+b}{2},$$

and

$$(2)\quad \left.\begin{aligned} x^2-3y&=1,\\ x-y^2&=1 \end{aligned}\right\}.$$

4. Find the sum of n terms of a geometrical progression whose first term is a and whose common ratio is r.

If r be <1, deduce the sum of an infinite number of terms of the progression.

If the pth term of an arithmetical progression be $\frac{1}{q}$ and the qth term be $\frac{1}{p}$, prove the sum of pq terms is $\frac{1}{2}(pq+1)$.

5. A traveller starts from A towards B, and at the same instant another traveller starts from B towards A. Two hours afterwards they meet at a point which is 24 miles from A, and the one arrives at A at the instant when the other is 20 miles from B. What is the distance between A and B, each traveller being supposed to go at a uniform rate?

6. Prove the truth of the Binomial Theorem for a positive integral exponent.

Show that the sum of the squares of the coefficients in the expansion of $(1+x)^n$, where n is an integer, is

$$\frac{|\underline{2n}}{\{|\underline{n}\}^2}.$$

7. Find the present value of £P due n years hence, compound interest being reckoned at the rate of £r per unit per annum.

Two equal sums of money are invested at different rates of compound interest. The sum which is invested at the higher rate doubles itself in p years, and that which is invested at the lower rate doubles itself in q years. Prove that in $\frac{pq}{q-p}$ years the amount of the former will be twice the amount of the latter.

8. Find the amount at the end of n years of an annuity of £P per annum, compound interest being reckoned at the rate of £r per unit per annum.

A company issues 5 per-cent debenture bonds to the amount of £1,000,000. It sets aside £60,000 per annum to pay the interest on the outstanding bonds and to gradually redeem them by annual drawings at par. Prove that the whole issue will be redeemed by the end of 37 years from the date of issue, given $\log 6 = \cdot 7781513$ and $\log 1\cdot 05 = \cdot 0211893$.

9. A cricketer, on getting out in the first innings of a match, finds that he has in that innings increased his average for the season by 2; on getting out in the second innings, he finds that he has still further increased his average by 2. If in the whole match he made 80 runs, find how many runs he made in each innings of the match.

10. Sum the series

(1) $1.2+3.4x+5.6x^2+\ldots$ to infinity, x being <1,

and (2) $2.5.8+5.8.11+8.11.14+\ldots$ to n terms.

11. A company of 24 persons is arranged at random in a straight line; prove that the chance of any three given persons of the company being consecutive is $\frac{1}{92}$.

12. In a right-angled triangle, prove that the square on the hypothenuse is equal to the sum of the squares on the two sides.

Show how to construct geometrically a straight line whose length is $\sqrt{5}$ inches.

Second Paper.

1. What is the use of the journal in book-keeping?

Mention the entries required on receipt of a remittance from an agent resulting in a credit balance. Assume that, when preparing the balance-sheet, all the agents had credit balances, and express the necessary journal entry.

2. The net income of an estate after deducting 12 per-cent for repairs, 3 per-cent of the remainder for collection, and then 8*d.* in the £ on the rest for income tax, is £1,547. 3*s.*: find the gross income.

3. A woman spent 10*s.* 10*d.* in buying eggs, some at two a penny and others at three for twopence; she sold them for £1, thus gaining a halfpenny on each egg; how many of each did she buy?

4. Prove that the quadratic equation $ax^2+bx+c=0$ can have only two roots, and find the condition that they may be both real.

If x be real, find the smallest positive value of $x+\frac{a^2}{x}$.

5. Find the number of permutations of n things taken altogether, there being p things of one kind, q of another kind, and the rest being all different.

Find the number of different words that can be formed from the letters in the word *repetition*, there being always a consonant both in the first and last place.

6. Find the nth term in the recurring series

$$2+17x+95x^2+461x^3+\ldots\ldots,$$

and sum the series to infinity, x being less than unity.

7. If m sovereigns in a row stretch as far as n pennies, and p sovereigns in a heap are as high as q pennies, compare the value of equal bulks of gold and copper, it being assumed that the area of a circle varies as the square of its radius and that the value of the copper in 240 pennies is equal to that of the gold in a sovereign.

8. If $x<1$, prove that

$$\log_e(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots\ldots \textit{ad inf.},$$

and show how the value of logarithms to base 10 may be calculated.

Prove that $\log_e(1+x-2x^2)=x-\frac{5}{2}x^2+\frac{7}{3}x^3-\frac{17}{4}x^4+\ldots$ *ad inf.*, and find the general term of the series.

9. The rental of a house under a lease which has 12 years to run (the rent having just been paid) is £52 and the actual net letting value is £65. If the old lease be surrendered and a new lease for 36 years be granted in its place, prove that an equivalent rental under the new lease is very nearly £58. 10*s.*, interest being reckoned at 4 per-cent per annum.

Given $\log 2=\cdot30103$, $\log 13=1\cdot1139434$, and $\log 1\cdot601=\cdot2044008$.

10. A throw is made with four dice, each die having six sides numbered 1, 2, 3, 4, 5, and 6 respectively. Prove that the chance that the sum of the readings of the dice is 16 is $\frac{125}{1296}$.

11. An event must have happened from one of n mutually exclusive causes the antecedent probabilities of which are $P_1 . P_2, \ldots P_n$ respectively; the probabilities that, when these causes exist, the event happens are respectively $p_1, p_2, \ldots p_n$; prove that, on any occasion on which the event happens, the probability that it was caused by the rth cause is $\dfrac{P_r p_r}{P_1 p_1 + P_2 p_2 + \ldots + P_n p_n}$.

The antecedent odds against an event happening is 1,000 to 1, but four witnesses, each of whom correctly states what he has seen on 9 occasions out of every 10, agree in stating that it happened. Prove that the chance that it did happen is a little greater than $\frac{13}{15}$.

12. Prove that angles in the same segment of a circle are equal.

P is any point on the arc of a segment of a circle whose chord is A B; the angles P A B and P B A are bisected by two straight lines which meet in O. As P moves on the given arc, find on what curve the point O moves.

Examination for Admission to the Class of Associate, (Part II).

Examiners—Messrs. A. W. Sunderland, Geoffrey Marks, H. C. Thiselton, and S. G. Warner.

First Paper.

1. Having given a table of v^n, how would you proceed to form a table of the Sinking Fund which, annually accumulated at the same rate of interest, would amount to 1 in n years? Explain by general reasoning, and without the use of symbols, the connection between the two functions.

2. Investigate a convenient formula for ascertaining approximately the true rate of interest yielded by terminable debentures issued at a premium and redeemable at par.

Apply the formula so obtained to determine the rate of interest yielded by a terminable 6 per-cent debenture, repayable at par at the end of 20 years, purchased for £119. 10*s*.

3. Investigate an expression for Q^1_{xyz} in terms of the expectations and probabilities of life.

4. Give a strictly theoretical formula for the office annual premium for a child's endowment, the premiums to be returned if the child dies, and to cease if the father of the child dies, before the policy matures.

What approximation would you use in practice?

5. Explain the symbol $\mathring{\hat{a}}_{y|x}^{(m)}$. State which you consider the best approximation to its value, and show how it is derived.

6. Explain what is meant by the hypothetical or re-assurance value of a policy.

Show under what conditions the net-premium value of a policy is less than, equal to, or greater than, the hypothetical value.

7. Obtain expressions for $A^{2}_{\underset{1}{x}yz}$, A^{2}_{xyz}, A^{3}_{xyz}, $A^{1}_{x:yz}$, and $A^{\;\;1}_{\overline{xy}:z}$.

8. Explain the two forms of the N commutation columns for single lives, and the advantages of each. Which form is the more suitable for select tables, and why? Give the formulas in commutation symbols for the value of an annuity on a select life, and for the value of an endowment assurance policy effected t years ago on a life then select.

9. Show how to calculate and verify a complete table of temporary life annuities by means of the arithmometer. Explain how such a table may be used to furnish annual premiums for pure endowments.

10. State concisely the provisions of "The Policies of Assurance Act, 1867." How were they affected by "The Customs and Inland Revenue Act, 1888"?

11. Discuss the eligibility for the purpose of life office investments of (1) Mortgage Bonds of American Railroads (sterling and currency); (2) Debentures of English Railway Companies; and (3) Preference Shares in Industrial Companies.

12. Discuss the question of allowing surrender-values to pure endowments and term assurances, and explain the apparent anomaly that, while it is not usual to allow such surrender-values in practice, they are freely granted when both benefits are combined in an endowment assurance.

Second Paper.

1. Obtain a formula for the number of years within which money will double itself at compound interest. State the approximation usually made in practice and show that the resulting error is nearly constant for all rates of interest.

2. Define clearly the terms "nominal rate of interest", "effective rate of interest", and "force of interest." Find the present value and amount of an annuity-certain for n years payable p times a year, interest at the nominal rate i being convertible q times a year, and deduce the present value and amount when interest is convertible momently.

3. A life office advances £1,000, repayable in 10 years, by an annuity to secure interest at the rate of 5 per-cent. and provide for the accumulation of the sinking fund at the rate of 3 per-cent. When the sixth annual payment becomes due, the borrower desires to cancel the arrangement and repay the loan at once. Find the amount of capital actually outstanding and state what sum you would advise the office to accept in satisfaction of its claim.

4. A widows' annuity society is recruited by the annual entrance of k married couples, the husband being aged x and the wife y. After n years from the commencement, how many widows will there be on the fund, and how many when the fund has existed long enough to have attained a stationary state?

5. Define the force of mortality and the central death rate, and write down the approximate relation between them. Show how, from a table of the latter function at all ages, to construct a mortality table.

6. How would you approximate to the value of an assurance payable on the death of the survivor of x and y, if he die before z?

7. Explain by general reasoning the equations

$$A_x = 1 - d(1 + a_x)$$

$$P_x = \frac{dA_x}{1 - A_x}$$

$$_nV_x = (P_{x+n} - P_x)(1 + a_{x+n})$$

8. Find a formula for the value of a reversionary annuity payable for the remainder of the life of B. after the death of A, the annuity being reducible by one-half should such death not occur for seven years, and by two-thirds should it not occur for 10 years from the present time.

9. Having at your disposal the results of two censuses of an increasing community, taken at an interval of 10 years, with a statement of the numbers dying annually in each year of age for the same period, show how to construct, as accurately as the data will allow, a table of mortality applicable to the community in question.

In connection with a scheme of National Pensions of 5*s.* a week to persons of 60 and upwards, show how to calculate the immediate annual cost to the nation of such a scheme and the probable cost 80 years hence.

10. Explain the meanings of the symbols D_x, N_x, S_x, M_x, R_x, and show how to construct tables of their values.

Interpret the expressions $\frac{N_{x-n} - N_x}{D_x}$ and $\frac{M_{x-n} - M_x}{D_x}$.

11. What is meant by conversion tables, and what are their principal uses? Explain the reason of the recurrence, in certain cases in *Orchard's Tables*, of the last three figures of the values of the single premiums.

Show how *Orchard's Tables* may be used in applying the bonus on a whole-life policy to convert the policy into an endowment assurance.

12. Discuss generally the desirability of making the truth of the answers in a proposal for life assurance and in the medical report a definite part of the basis of the contract.

13. What precautions should be observed in (*a*) lending upon leasehold property; (*b*) investing in ground rents?

Examination for Admission to the Class of Fellow (Part III, Section A).

Examiners—Messrs. George King, M. N. Adler, F. E. Colenso, Wm. Hughes, L. M. Simon, and Geo. Todd.

First Paper.

1. A life assurance company is about to be formed. What legal requirements have to be complied with before business can be commenced?

2. A B produces to a life assurance company a policy effected by X Y on his own life, with proof of the death of X Y. The policy is endorsed, "Pay to the order of A B", which endorsement is dated and signed by X Y. What should the company require before paying the claim? Does the domicile of either A B or X Y affect the case?

3. Give an epitome of a settlement of a life policy made by a man in contemplation of marriage.

4. Application is made to an assurance company for the payment of the sum assured by a policy duly kept on foot, on the ground that the person whose life is assured has disappeared and has not been heard of for a considerable period. State in detail what enquiries you would make, and what evidence you would require.

5. Give the leading points in the Bank Act of 1844, and exhibit one of the weekly returns of the Bank of England.

How would you be able to tell whether a change in the bank rate is likely to take place?

6. State what you know of the recent stoppage of Australian banks, and of the different measures taken on their reconstruction with regard to their deposits.

7. What points is it necessary to consider and provide for in granting policies payable in silver currency? Draft for insertion in a policy clauses suitable to the circumstances.

8. Explain Lagrange's Theorem for Interpolation, and show how by its means, coupled with a continued method, to complete by differences a table of a function from certain given values which are not equidistant.

Having given $\log l_{40}=4{\cdot}91528$, $\log l_{44}=4{\cdot}89676$, $\log l_{47}=4{\cdot}88069$, and $\log l_{52}=4{\cdot}84793$, complete the table from $\log l_{40}$ to $\log l_{52}$ inclusive, by a constant third difference.

9. What are the objects aimed at in graduating a mortality table, and what are the conditions of a good graduation?

10. It was claimed by Woolhouse for his graduation formula obtained by the method of central differences, that it produces in the immediate neighbourhood of the graduated point a curve of the third order. Explain the ground for this claim, and state whether you consider it well founded.

11. A company closes its books annually on 30 September. The mortality experience is to be extracted, and the following are the data which alone can be supplied, namely—

(*a*) Calendar year of birth.
(*b*) Financial year of entry.
(*c*) Financial year of exit.
(*d*) Financial year of death.

Give a brief but complete statement of the method you would employ to prepare an ungraduated mortality table.

12. In the case of a society in which the rate of withdrawal of members depends on the age and not on the duration of membership, how would you construct a combined mortality and withdrawal table, to show l_x, the number of members remaining at age x; d_x, the number of members dying between ages x and $x+1$; and w_x, the number of members withdrawing between ages x and $x+1$; having given the exact date of birth, and the calendar years of entry, of death, and of withdrawal, of members.

Second Paper.

1. What is an industrial insurance company, and in what respect does the legislation as regards industrial insurance companies differ from that relating to ordinary friendly societies?

2. What is an interpleader suit? and what is meant by a garnishee order?

3. State generally the principal alterations affecting the death duties, introduced by the Finance Act of 1894, and draft a form of endorsement making an existing policy available for the payment of estate duty before grant of probate.

4. Give a general account of the provisions of "The Life Assurance Companies Act, 1870", including the schedules of returns to be made to the Board of Trade.

What alterations, if any, would you recommend in the Act, and in the Schedules, in the event of the introduction of a Bill to amend the Act?

5. Describe the general purpose and effect of "The Policies of Assurance Act, 1867." In what way did it alter the previously existing law?

6. State your opinion as to the causes which have led to the reduction in the rate of interest obtainable on first-class investments.

7. What steps have been taken in recent years to reduce the National Debt of the United Kingdom, or in other ways to lighten its burden?

What is meant by Exchequer Bills, and what object is served by their issue?

8. State what you know of the laws and relations of the symbols

$$\mathrm{D},\ \Delta,\ \text{and}\ \frac{d}{dx}.$$

D being defined by the relation $\mathrm{D}u_x = u_{x+1}$.

9. Explain the method of "Integration by parts" as applied to the finite integration of functions.

Apply the method to the finite integration of functions of the form $a^x . \phi(x)$, where $\phi(x)$ is rational and integral.

Find by integration the sum of n terms of the series

$$1.2.a+2.3.a^2+3.4.a^3+4.5.a^4+\&c.$$

10. Explain the various methods that have been suggested for calculating the numerical values of the constants in Makeham's method of graduating mortality tables.

11. How would you set about graduating a mortality table by a graphic method? What function would you choose for graduation, and what are the merits and demerits of this method as compared with a system of numerical distribution of irregularities, such, for instance, as Woolhouse's?

12. Explain the "Nearest Duration" method of constructing mortality tables.

Can you suggest any plan of tabulation under this method, so that it may be possible in further investigations to assign each withdrawal accurately to its proper insurance year?

Examination for Admission to the Class of Fellow (Part III, Section B).

Examiners—Messrs. George King, M. N. Adler, F. E. Colenso, Wm. Hughes, L. M. Simon, and Geo. Todd.

First Paper.

1. Policies are granted under a table of premiums reduced by anticipation of future bonus. How should such policies be treated for valuation purposes and for surrender? Discuss the effect of early lapsing or surrender in the case of such policies.

2. It appears desirable to alter the basis of valuation of a Life Office to one requiring larger reserves. Indicate the chief points to be considered and treated of in a report to the Directors advising the alteration, in the case of, (*a*), a Mutual Company, and, (*b*), a Proprietary Company.

3. Having given a table showing the numbers living and remaining members of a Society at each age, and the numbers in each year of age ceasing to be members (*a*) by death, and (*b*) by withdrawal, respectively, how would you make a valuation (allowing for withdrawals) of the Society, which gives (1) a pension on attaining age z, (2) a return of double the contributions paid should death occur before age z, and (3) a return of the exact contributions paid should withdrawal take place before age z?

To simplify the question assume that all members pay the same contribution irrespective of age at entry, and draw the same pension on reaching age z.

4. In valuing a reversion for the purpose of loan or purchase, how would you purpose to deal with the estimated realizable value of the funds in the case of leasehold securities, ground rents, securities not fully paid up, and securities in which there is not a free market?

5. Discuss the difference in principle between the formulas given by Mr. Jellicoe and Dr. Sprague, respectively, for the valuation of reversionary interests, and state which you would consider preferable in the case of a Company which invests largely in the purchase of such interests.

What considerations would guide you in determining the amount which could be advanced on mortgage of a well-secured contingent reversion, if there were no guarantee for the payment of premium and interest during the continuance of the joint lives?

6. Define a Distringas, and a Stop Order. In what way do they operate as a safeguard to a lender on security of Stock?

7. Explain how a Life Company is assessed to Income Tax. What special questions arise in the case of Companies transacting Annuity business?

8. A writer in a Medical paper states that his enquiries among 737 phthisical patients showed that 351 cases arose from among 1,041 children of phthisical parents, and 386 cases from among 1,552 children of non-phthisical parents; and from this he infers that "the influence of heredity cannot be put higher than 8·8 per-cent of "cases amongst the children of phthisical parents in excess of the "cases amongst children of non-phthisical parents."

Is it possible that he mis-states the influence of heredity, and, if so, in which direction is it probable that his error lies? What data would you consider necessary for a satisfactory investigation of the question?

9. Determine by aid of the Integral Calculus the correction necessary to derive $\mathring{a}_x$ from a_x.

10. Show how to obtain formulas for the approximate evaluation of integrals between given limits.

Calculate an approximate value for $\bar{a}_{60}$ by means of the following data:

$$_{12}p_{60}v^{12} = \cdot354,\ \ _{24}p_{60}v^{24} = \cdot045,\ \ _{36}p_{60}v^{36} = \cdot000.$$

11. Upon the death of a lunatic lady intestate, certain personal estate will be divisible according to the Statutes of Distribution. The lunatic is a widow, with children, some of whom desire to raise a loan upon security of their expectations. Specify the various circumstances which, in your opinion, would enable a lender to obtain a perfect security.

12. An office advances £11,250, this sum including an estimate of £250 for costs, in consideration of the sale of an annuity charged upon the reversionary life interest, in an estate of ample value, of a gentleman aged 38, expectant upon the death without issue of his uncle aged 54. The reversioner's life is insurable at the rate of £2. 13*s*. 9*d*. per-cent per annum, and the insurance against risk of issue can be effected at a single premium of £8 per-cent.

Find the amount of the reversionary charge per annum on the assumption that the purchaser sets up annuities on the books to provide the renewal premiums during the joint lives and interest at $4\frac{1}{2}$ per-cent on the total outlay, the annuity-value employed in the one case being $a_{38:54} = 10\cdot575$, and in the other $\mathring{a}^{(2)}_{38:54} = 10\cdot992$. Give the amount of the redemption-money, and set out the transaction in such a way as to test the accuracy of your work.

Second Paper.

1. An Office distributes its surplus as a uniform percentage per annum (counting from the last preceding valuation) on the sums assured and existing bonus additions. How would you provide for the future maintenance of the rate of bonus, if the business on the books is increasing, stationary, or decreasing?

What process would you follow to distribute a given surplus?

2. Describe a "Select Mortality Table." How would you use one in the valuation of the ordinary whole-life policies of an Insurance Company.

(*a*) If the valuation is to be made in groups of policies, and

(*b*) If each policy is to be valued separately?

3. A Company makes its valuations by the net premium method at a rate of interest considerably lower than that earned on its funds. From time to time it has changed its tables of premiums for participating policies, so that it has on its books policies at various rates of premium. What method of distribution of surplus would you recommend, giving your reasons, and showing how you would carry out the method practically?

4. A Life Office distributes its surplus as a uniform percentage per annum, since the last preceding investigation, on the sums assured, and gives an interim bonus in the case of those policies which take their share as a reversion, but not in the case of those which take their share as reduction of premium. How would you adjust the reductions of premium so as to deal fairly with all parties,

(*a*) When the interim bonus is equal to the full periodical bonus, and

(*b*) When the interim bonus is 80 per-cent of the periodical bonus?

5. An office which values its liabilities at 3 per-cent and states its assets in the balance sheet at their cost price, holds several $3\frac{1}{2}$ per-cent securities, bought at par, which now stand in the market at 115. It is suggested that these investments should be sold to secure the profit, and that the difference realized should be carried to an Investment Reserve Fund. Do you recommend this course, and what considerations influence your opinion?

6. Mention the principal methods which have been suggested for apportioning a trust fund between the Life Tenant and the Reversioner by mutual consent, and describe fully the method which you would select, giving your reasons for adopting it.

How would your conclusions be modified if the apportionment were to be made at the instance of

(*a*) The Reversioner, or

(*b*) The Life Tenant?

7. In some Companies it is the custom to grant policies on under-average lives at the normal rate of premium for the actual age, but with a condition that a deduction is to be made from the sum assured in the event of early death, such deduction to diminish as the age increases. Give your opinion generally on this method of treating under-average lives. Are there any cases in which it is more suitable than in others?

8. What are the advantages and disadvantages attaching to each of the following schemes for providing National Old-Age Pensions?

(*a*) That of accumulating a fund derived from annual contributions, either wholly or partly provided by the State, actuarially calculated to supply a pension to all who attain a certain age.

(*b*) That of raising annually by taxation an amount sufficient to supply the pensions of the year out of revenue.

9. What are points of contrary flexure in a curve; and what change do they indicate in the relation of the values of the ordinates?

Draw roughly the form of curve exhibited when the values of the ordinates proceed as follows:

	1st difference	2nd difference
(*a*)	positive	positive.
(*b*)	positive	negative.
(*c*)	negative	negative.
(*d*)	negative	positive.

10. Find the form taken by l_x when the force of mortality is composed of three parts, namely, a constant, an arithmetical progression, and a geometrical progression; that is when

$$\mu_x = A + Bx + Dc^x.$$

The Candidate having handed in his answers to the foregoing questions, will be supplied with the following books:—"Institute of Actuaries' Life Tables"; a Premium Conversion Table (Orchard or Rothery and Ryan); and a book of logarithms; so that he may give numerical solutions to the following questions. The Institute of Actuaries' Tables are supplied for the sake of convenience, and it must not be inferred that they are held to be suitable. The Candidate is requested to state in answering each of the two questions what mortality tables and rates of interest he would consider most suitable for adoption in such a case in practice.

11. Landed estates producing a net rental from agricultural holdings of £6,000 per annum clear of all deductions, stand limited to trustees in trust to pay annuities thereout of £2,000, £1,000, and £1,000, to three ladies, A aged 65, B aged 70, and C aged 75, for their respective lives; the surplus income being payable as to one half thereof to A for her life, as to one fourth to

B for her life, and as to one fourth to C for her life; and, subject to such annuities and life interests, the estates are held in trust for C and her heirs. What in your opinion is the largest sum that could be safely advanced upon the mortgage of C's reversionary interest?

12. An officer in the army, aged 28, is entitled to a reversionary life interest in the personal estate of his grandfather, contingent upon his surviving his father aged 54. The estate consists of

£5,000 Consols,	$2\frac{3}{4}$	per cent.	
5,000 ,,	$2\frac{1}{2}$	,,	
2,000 Debentures,	4	,,	and repayable in 1904.

At what price would you value this interest for purchase by an Assurance Company? What enquiries would you make in the first place before entertaining the offer?

PROCEEDINGS OF THE INSTITUTE.—Session 1894–95.

First Ordinary Meeting, 26 *November* 1894.

The first ordinary meeting of the session 1894-95 was held at the Hall of the Institute, on the 26th day of November 1894.

The President (Mr. A. J. Finlaison) in the Chair.

The President delivered an inaugural address.

Second Ordinary Meeting, 17 *December* 1894.

The President (Mr. A. J. Finlaison) in the Chair.

A ballot was taken for the election, as Honorary Member, of Lieut.-Colonel W. H. Oakes. Messrs. H. E. Nightingale and T. J. Searle acted as Scrutineers, and Colonel Oakes was declared duly elected.

A ballot was next taken for the election, as Corresponding Member, of Mons. Amédée Bégault, the same gentlemen acting as Scrutineers. M. Bégault was declared duly elected.

A paper "On an Investigation of the Mortality and Marriage Experience of the Widows' Funds of the Scottish Banks", by Messrs. Archibald Hewat and James Chatham, was read by Mr. Hewat.

The following gentlemen took part in the discussion:—Messrs. G. F. Hardy, Ryan, R. P. Hardy, G. King, Nash, and the President.

Third Ordinary Meeting, 28 *January* 1895.

The President (Mr. A. J. Finlaison) in the Chair.

A paper entitled "On some Considerations suggested by the large Number of the Existing at the close of the Institute Observations of 1863", was read by the author, Mr. R. R. Tilt.

The following gentlemen took part in the discussion:—Messrs. Wyatt, Whittall, G. F. Hardy, Pulley, Higham, Manly, and the President.

Fourth Ordinary Meeting, 25 *February* 1895.

The President (Mr. A. J. FINLAISON) in the Chair.

Mr. G. H. Ryan read a paper entitled

"I. On the standard of Solvency in Life Assurance Companies as affected by guaranteed benefits.

"II. On certain Methods of Reconstructing an insolvent Life Assurance Company."

An account was afterwards given by Mr. Ryan of a recent visit to Canada and the United States.

The following gentlemen took part in the discussion:—Messrs. Manly, Bailey, R. P. Hardy, Coles, Nash, Adler, and the President.

Fifth Ordinary Meeting, 25 *March* 1895.

The President (Mr. A. J. FINLAISON) in the Chair.

A ballot was taken for the election, as Associate, of Mr. Donald Macphail, F.F.A. Messrs. H. C. Thiselton and F. Bell acted as Scrutineers, and Mr. Macphail was declared duly elected.

A ballot was next taken for the election, as Associate, of Mr. Henry Moir, F.F.A., the same gentlemen acting as Scrutineers. Mr. Moir was declared duly elected.

A paper entitled "On the Distribution of the Divisible Surplus of a Life Assurance Company, with special reference to the Method originated by Dr. Sprague and other Methods derived therefrom", was read by the author, Mr. G. J. Lidstone.

The following gentlemen took part in the discussion:—Messrs. G. King, G. F. Hardy, Bailey, Browne, Chisholm, Manly, and the President.

Sixth Ordinary Meeting, 29 *April* 1895.

The President (Mr. A. J. FINLAISON) in the Chair.

The following papers were read by their respective authors:

"On a method of Computing the Temporary Deductions to be made from the Sums Assured, upon Rated-up Lives, in lieu of Extra Premium", by Mr. G. F. Hardy.

"On a New Method of performing approximately certain Operations in Multiplication and Division", by Mr. J. A. Robertson.

The following gentlemen discussed the subjects treated of in the papers:—Messrs. Chisholm, S. Day, Sutton, Burridge, and the President.

The Forty-Eighth Annual General Meeting, 8 *June* 1895.

The President (Mr. A. J. FINLAISON) in the Chair.

The proceedings at the Annual General Meeting will be found on page 148.

REPORT, 1894–95.

"The Council have pleasure to report to the members upon the progress of the Institute during the session of 1894-95, the forty-seventh year that it has been in existence.

"The increase in the number of members in the year has been 41, as

compared with 60 in that which preceded it. At the end of the year in which the Institute was incorporated by the Royal Charter the number of members was 434, while five years later, at 31 March 1890, it was 601. Since that time it has grown as follows:

At 31 March 1891 to 620,
,, 1892 ,, 645,
,, 1893 ,, 674,
,, 1894 ,, 734,
,, 1895 ,, 775.

"The following schedule shows the additions, changes, and losses in the membership, which have occurred during the year ending 31 March last.

Schedule of Membership, 31 *March* 1895.

	Honorary Members	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1894 .	1	169	208	345	11	734
ii. Withdrawals by						
(1) Death . . .	...	4	1	1	...	55
(2) Resignation . .	...	1	4	23	...	
(3) Default in payment of Subscriptions . .	...	1	1	19	...	
	1	163	202	302	11	679
iii. Additions to Membership						
(1) By Election . .	1	...	3	...	1	96
(2) By Order of Council	..	..	...	84	...	
(3) By Re-instatement	..	1	...	5	1	
	2	164	205	391	13	775
iv. Transfers						
(1) By Examination: *from* Associates	..	...	6	...	...	...
to Fellows .	...	6	...	...	...	..
	2	170	199	391	13	775
(2) By Examination: *from* Students	...		...	3	...	...
to Fellows .	...	3	..	...	...	...
	2	173	199	388	13	775
(3) By Examination: *from* Students	...	...	...	22	...	...
to Associates .	...	..	22	...	...	...
v. Number of Members in each Class on 31 March 1895 .	2	173	221	366	13	775

"The Council have, with great regret, to report the loss by death during the year of four Fellows, namely: Mr. W. S. Gover, Mr. A. W. Sunderland, Mr. James Taylor, and Mr. James Valentine.

"Mr. Sunderland, to whom the Institute is indebted for valuable contributions to the *Journal*, was a Member of Council at the time of his death. Mr. Valentine, formerly a Member of Council, was also a contributor to the *Journal*.

"The vacancy in the Council caused by the death of Mr. Sunderland was filled up in the prescribed manner by the election of Mr. M. N. Adler.

"The Accounts for the year will be seen to be satisfactory, the total amount of funds now being £5,605. 8*s*. 9*d*., showing an increase in the year of £470. 2*s*. 7*d*.

"The Revenue Account and Balance Sheet are given herewith (p. 147).

"The Annual Subscriptions, together with admission and other fees, amounted to £1,573. 19*s*. 0*d*., showing an increase of £100. 16*s*. 0*d*. over those of the previous year.

"The total Income for the year was £1,969. 10*s*. 9*d*., and the total Expenditure £1,499. 8*s*. 2*d*.

"The stock in hand of the Institute publications at date is as follows:

No. of Copies	Description of Work
178	*Text-Book*, Part I.
421	,, ,, II.
84	Mortality Experience Tables.
907	Government Joint-Life Annuity Tables.
563	Logarithm Cards.
403	Messenger Prize Essay (Friendly Societies).
512	Index to 10 Vols.
87	,, to 20 ,,
9,284	Parts of *Journal*.

"The following papers were submitted at the sessional meetings of the Institute, namely:

"26 *November* 1894—An Address by the President, Mr. A. J. Finlaison, C.B.

"17 *December* 1894—'On an Investigation of the Mortality and Marriage Experience of the Widows' Funds of the Scottish Banks', Messrs. Archibald Hewat and James Chatham.

"28 *January* 1895—'On some Considerations suggested by the large Number of the Existing at the close of the Institute Observations of 1863', Mr. R. R. Tilt.

"25 *February* 1895—'I. On the standard of Solvency in Life Assurance Companies as affected by guaranteed benefits.'

'II. On certain Methods of Reconstructing an Insolvent Life Assurance Company', Mr. Gerald H. Ryan.

"25 *March* 1895—'On the Distribution of the Divisible Surplus of a Life Assurance Company, with special reference to the Method originated by Dr. Sprague, and other Methods derived therefrom,' Mr. G. J. Lidstone.

"29 *April* 1895—'I. On a method of Computing the Temporary Deductions to be made from the Sums Assured, upon Rated-up Lives, in lieu of Extra Premium', Mr. G. F. Hardy.

'II. On a New Method of performing approximately certain Operations in Multiplication and Division', Mr. J. A. Robertson.

"For the Examinations held in the United Kingdom on 19, 20, 22 and 23 April last, 177 candidates presented themselves, namely:

96 for Part I.
45 ,, ,, II.
21 ,, ,, III, Section A.
15 ,, ,, III, ,, B.

"Of these the following numbers were successful:

51 in Part I.
12 ,, ,, II.
13 ,, ,, III, Section A.
4 ,, ,, III, ,, B.

"The following are the successful candidates, the names in each class being arranged alphabetically.

PART I.

Class I:

W. A. Basham.
H. E. W. Lutt.
S. Macnaghten.
Thos. Peele.
W. Thistlewaite.

Class II:

H. T. Adlard.
S. Adlard.
A. G. Challen.
C. R. V. Coutts.
F. A. Dawson.
J. Goodwyn.
B. T. Ham.
W. M. Haycraft.
A. J. Hicks.
G. H. Lawton.
P. C. Mayhew.
A. Moorhouse.
W. E. Norton.
H. J. Rietschel.
A. J. Sindall.
H. C. Wyley.

Class III:

F. L. Collins.
F. R. Dickinson.
H. Dougharty.
W. P. Elderton.
W. L. Gledstone.
F. S. Goggs.
A. Golding.
W. H. Grainger.
H. Harris.
D. McK. Hopping.
C. E. Howell.
F. C. Hudson.
H. A. Hurst.
C. S. Kelham.
W. Longstaff.
B. May.
G. L. Moore.
C. Nelson.
C. E. Pawlett.
H. C. Powell.
C. H. Salter.
N. S. Sandeman.
W. M. Simpson.
A. E. M. Smith.
L. B. Tappenden.
J. T. Taylor.
T. P. Wansbrough.
R. A. H. Watson.
S. L. Welch.
F. J. Woods.

Part II.

Class I:

J. Spencer.

Class II:

W. T. Butterfield.
L. T. Heness.
O. Kentish.

Class III:

A. C. R. Cockman.
H. W. Glover.
L. K. Pagden.
M. Rees.
R. G. Salmon.
G. M. Searle.
W. R. Strong.
L. A. Wintle.

Part III.—Section A.

Class I:

*A. R. Barrand.

Class II:

H. J. Baker.
A. D. Besant
N. Miller.
H. A. Thompson.

Class III:

R. W. Barton.
A. H. Clarke.
C. P. Dawson.
*F. W. Fulford.
H. N. Sheppard.
W. A. Sim.
L. Stahlschmidt.
W. Worthington.

Those marked * passed in both Sections.

PART III.—SECTION B.

Class I:

*A. R. Barrand.

Class II:

J. N. Lewis. J. Watson.

Class III:

*F. W. Fulford.

Those marked (*) passed in both sections.

"In the Colonies the Examination entries numbered 50, as under:

For Part	I,	30.
,,	II,	17.
,,	III, Section A,	2.
,,	III, ,, B,	1.

"The results of the Colonial Examinations will be duly announced.*

"The Tables of Joint-Life Annuities calculated in 1884 upon the observations of the mortality of Government Annuitants—which, as announced in the Report for 1893-94, had been placed at the disposal of the Institute—were published just before the close of the year 1894-95, and will, it is hoped, prove of service to the profession and the public.

"A new Catalogue of the Library was published in January last, a copy of which has been sent to each member of the Institute.

"The Council have pleasure in reporting that the French Edition of the *Text-Book*, Part II, has now been published, and that a copy of the work, kindly presented by the Translators, is in the Library of the Institute.

"The Council have pleasure also in reporting that Dr. Sprague has acceded to a suggestion that the Institute should publish certain Tables based upon his Select Mortality Experience investigations, founded on the Tables published in the Institute volume of 1869, and that the work is now in progress.

"As announced by the President at the last Annual Meeting, and notified by Circular to each Member in July last, the Council has offered from the Messenger Legacy Fund two Prizes, of the value of 50 guineas and

* See p. 152.

25 guineas respectively, for the two best Essays on the following subject, namely:

"'The Books and Forms to be used in scheduling the particulars of the risks of a Life Assurance Company under its Assurance and Annuity Contracts, for periodical or interim Valuations, Distribution of Surplus, and for Investigation of the Rates of Mortality, Surrender and Lapse.'

The Essays are to be sent in to the Honorary Secretaries not later than 30 September 1895.

"The Institute has received an invitation to take part in a Congress of Actuaries to be held in Brussels in September next, with the object, principally, of securing the adoption of a uniform system of notation. The Council has decided to express the Institute's approval of the project, and the following gentlemen have consented to attend the Congress as Official Representatives of the Institute: The President, Mr. A. J. Finlaison, Mr. Aug. Hendriks, Mr. George King, and Mr. G. H. Ryan.

"Members of the Institute of Actuaries are invited by the Organizing Committee of the Congress to attend it as members, and the Assistant Secretary of the Institute will, on application before 1 August, supply information as to the necessary procedure.

"The Council have also pleasure in referring to the expenditure on account of an Index to the ten volumes of the *Journal* from the twentieth to and inclusive of the thirtieth of the series, which is in course of preparation under the superintendence of Mr. G. H. Ryan, the Honorary Editor.

"In connection with the proposed new Mortality Experience Investigation by the Institute, in conjunction with the Faculty of Actuaries in Scotland, the Council have to report that the issue of cards to the contributing Offices was completed in December last, as far as Assured Lives are concerned, while those for the purpose of the Annuity Experience were issued in January. Several Offices have already returned the cards issued to them completed by the insertion of the desired information from their records.'

Dr. *Revenue Account for Year ending* 31 *March* 1895. Cr.

1894-95.	£ s. d.	£ s. d.	£ s. d.
Amount of Funds, 31 March 1894—			
Messenger Legacy Fund		303 1 1	
Brown Prize Fund		219 12 1	
Library Fund		298 5 0	
General Fund		4,314 8 0	5,135 6 2
Annual Subscriptions—			
Town Fellows	195 6 0		
Country ,,	67 4 0		
Fellows admitted since 1884:			
(1) By Examination	166 19 0		
(2) By Ballot	31 10 0	460 19 0	
Town Associates	182 14 0		
Country ,,	72 9 0		
Associates admitted since 1884:			
(1) By Examination	98 14 0		
(2) By Ballot	33 12 0	387 9 0	
Students		393 15 0	1,242 3 0
One Annual Subscription compounded for			10 10 0
Entrance Fees—			
Associates		4 4 0	
Students		111 6 0	115 10 0
Examination Fees			205 16 0
Sales of Publications—			
Journal		108 9 9	
Text-Book, Part I		31 13 0	
,, ,, II		89 13 3	
Log. Cards		2 14 10	
Mortality Experience		2 18 8	
Life Tables		15 12 0	
Hardy's Friendly Societies		0 5 5	
Government Annuity Tables		1 4 0	
		252 10 11	
Less Cost of Binding		26 4 11	226 6 0
Received for Use of Hall			26 5 0
Dividends and Interest			143 0 9
			£7,104 16 11

1894-95.	£ s. d.	£ s. d.
Journal—		
Cost of Nos. 173, 174, 175, and 176	346 18 9	
Clerical Assistance	12 12 0	
,, ,, on account of new Index to last 10 Vols.	20 0 0	379 10 9
Library—		
Binding and Purchases	50 12 9	
Balance of Cost of new Catalogue	172 2 10	222 15 7
General Expenditure—		
Rent	260 0 0	
Salaries (including Income Tax)	254 10 0	
Lecturers	52 10 0	
Examination Charges	64 7 4	
Meetings	19 4 1	
House Expenses (including Furniture)	37 18 10	
Corporation Duty	5 14 7	
Fire Insurance	13 8 0	
Stationery and Printing	100 16 1	
Postage and Telegrams	30 5 8	
Mortality Experience Investigation	47 12 0	
Paid on account of Government Annuity Tables	4 15 9	
Sundries	5 19 6	897 1 10
Funds 31 March 1895—		
Messenger Legacy Fund	308 14 1	
Brown Prize Fund	227 6 9	
Library Fund	298 5 0	
General Fund	4,771 2 11	5,605 8 9
		£7,104 16 11

Examined and found correct, 3 *May* 1895.

J. W. Miller,
R. M. Moore, } *Auditors.*
E. H. Holt,

Balance Sheet, 31 *March* 1895.

LIABILITIES.	£ s. d.	£ s. d.
Messenger Legacy Fund (£211. 1s. 10*d.* Consols), cost	203 17 8	
Accumulated Dividends	104 16 5	308 14 1
Brown Prize Fund (£200 Metropolitan Railway 4 per-cent Debenture Stock), cost	200 0 0	
Accumulated Dividends	27 6 9	227 6 9
Library Fund		298 5 0
General Fund		4,771 2 11
		£5,605 8 9

ASSETS.		£ s. d.
Consols (£2,490. 14s. 9*d.*), cost		2,386 10 0
Metropolitan Railway 4 per-cent Debenture Stock (£1,050), cost		1,185 11 3
Great Eastern Railway 4 per-cent Debenture Stock (£800), cost		1,031 10 3
Great Northern Railway Preferred Ordinary Stock (£400), cost		464 19 6
On Current Acct. at the London and Westminster Bank	£498 19 5	
Less Outstanding Cheque	14 11 8	484 7 9
Arrears of Subscriptions		52 10 0
		£5,605 8 9

Examined and found correct, 8 *May* 1895.

J. W. Miller,
R. M. Moore, } *Auditors.*
E. H. Holt,

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Annual General Meeting of the members was held at Staple Inn Hall, on Saturday, 8 June, the President, Mr. A. J. Finlaison, C.B., in the chair.

The Report of the Council (given on p. 140) having been read,

The PRESIDENT (Mr. A. J. Finlaison, C.B.), in proposing the adoption of the Report, said that the Report announced the continued prosperity of the Institute, the progressive increase in the number of members, the growth of the income, and, he was glad to say, of the expenditure, as this had been incurred on the worthy object of the library. The number of candidates who had presented themselves for examination had again increased this year, and it was very satisfactory to be able to report that many excellent papers were submitted. Congratulations were especially due to Mr. Barrand, who had this year repeated the success of Mr. Todhunter of last year, by attaining first class in both sections of the final examination. The Report recorded the death of four Fellows of the Institute, whose loss they would all deplore. He would refer more particularly to two of them, who had taken an active interest in the welfare of the Institute. Mr. Valentine formerly served upon the Council. Mr. Sunderland was a member of the Council at the time of his death, and had acted on many occasions as an examiner for the Institute, and contributed valuable and well-known papers to the *Journal.* The vacancy caused upon the Council was filled, according to the requirements of the Bye-Laws, by the election of Mr. Adler. A new catalogue of the library was issued in January last, for which thanks were due to the Chairman and members of the Library Committee, and especially to the Honorary Librarians. A volume of tables of joint-life annuities, founded upon the duration of the lives of Government annuitants, had been published at the expense of the Institute, and it was hoped that a collection of the "select" mortality experience tables, framed from the H^M Tables by Dr. Sprague, whom the Council thank for permission to publish his work, would soon follow. An index to vols. xxi to xxx of the *Journal* was in a forward stage of preparation, under the supervision of the honorary editor, Mr. Ryan, and would be issued to the members as soon as the careful and thorough way in which the work was being executed would permit. The French edition of Part II of the *Text-Book of the Institute* had been published, and the masterly way in which the translation had been executed was likely to aid in spreading a knowledge of actuarial science and of this Institute in the countries where the French language was current. A congress of actuaries had been appointed to be held in Brussels in the early part of September next, when distinguished members of the profession, from all parts of the world, were expected to be present. The Institute and the Faculty of Actuaries in Scotland had both expressed their adherence to the objects of the congress, and he was authorized to state that all members of the Institute would be welcome to attend. The subscription, which would entitle contributors to a copy of the proceedings, had been fixed on a reasonable scale, namely, at 20 francs—that is to say, at 16*s*. This subscription could be paid at the hall by those members who wished to join the congress. The Council were glad to be able to acknowledge the

receipt of several sets of cards contributed for the purposes of the new mortality experience investigation. He (Mr. Finlaison) then formally moved "That the Report and Accounts be received and adopted."

Mr. R. P. Hardy, in seconding the motion, referred to the educational work of the Institute, and to the great spread of technical knowledge involved in the examinations, and said that if not one of the 30 volumes of the *Journal* and no *Text-Book* had been created, and if no meetings had been held, this great and prevailing work could be allowed to be pleaded in bar of that judgment that went against all human efforts. In alluding briefly to the production of the new library catalogue, the extended index of the *Journal* in preparation, and to the publication at the common expense of the most useful table of joint-life annuities, he said that he claimed for the Institute—whether looking at the encouragement and advance given to actuarial work, or to its educational work, or the liberality in which it employed its corporate funds—that it was answering the purpose for which it was called into being, and that it had satisfied the anticipations of all who were associated with it.

The resolution was unanimously adopted.

Messrs. Searle and Justican having been appointed Scrutineers, a ballot was taken for the election of President, Vice-Presidents, Council, and Officers for the ensuing year. The Scrutineers reported that the following gentlemen, recommended by the Council, had been unanimously elected:

President.

Alexander John Finlaison, C.B.

Vice-Presidents.

Ralph Price Hardy.
Charles Daniel Higham.
George King.
Henry William Manly.

Council.

Alfred Barton Adlard.
Marcus Nathan Adler, M.A.
*Henry Walsingham Andras.
Arthur Hutcheson Bailey.
Arthur Francis Burridge.
James Chisholm.
Henry Cockburn.
Francis E. Colenso, M.A.
Alex. John Finlaison, C.B.
*Niel Ballingal Gunn.
George Francis Hardy.
Ralph Price Hardy.
Augustus Hendriks.
Charles Daniel Higham.
George King.
*Francis Laing.
Alex. George Mackenzie.
Henry William Manly.
Benjamin Newbatt.
Gerald Hemmington Ryan.
Frederick Schooling.
Louis Michael Simon.
Thomas Bond Sprague, M.A.
William Sutton, M.A.
John Bell Tennant.
*Herbert Cecil Thiselton.
Samuel George Warner.
Ernest Woods.
Frank Bertrand Wyatt.
Thomas Emley Young.

* New Members of Council.

Treasurer.

James Chisholm.

Honorary Secretaries.

Henry Cockburn.
Frank Bertrand Wyatt.

The President returned thanks on behalf of the Council, the Vice-Presidents, and Officers. He also thanked the meeting for having again elected him as President, and said that his endeavours, and those of his colleagues, would be to maintain and advance the position of the Institute of Actuaries amongst the scientific societies of the Metropolis.

Mr. Churchward proposed that Messrs. Moore, Holt, and Jellicoe, be elected Auditors for the ensuing year.

Mr. Roberts seconded the resolution, which was unanimously agreed to.

Mr. W. Hughes proposed a vote of thanks to the President, Vice-Presidents, Council, Officers, and Examiners for their services during the past year.

Mr. H. W. Andras seconded the motion, which was cordially agreed to.

The President, in acknowledging the vote on his behalf, and on behalf of the Vice-Presidents, Council, Treasurer and other Officers, said he was specially gratified to welcome there that day the President of the Actuarial Society of America, who had been a member of the Institute for upwards of 20 years. Mr. Emory McClintock's name was well known to all present, as a contributor to the *Journal*, and as the author of the valuable and successful essay "On the Effects of Selection."

Mr. E. McClintock said he accepted the welcome accorded to him personally with the greatest possible pleasure, and officially with the highest sense of gratification. His name had been on the List of Fellows for more than 21 years, although he had never been present at any of the meetings previously. This particular occasion of the annual meeting was to him specially gratifying, because it was not only in one sense the most important meeting of the year, but because he had hardly hoped to be able to get there, his steamer having only sailed from New York on the 30th of May. He felt a little embarrassment in voting for the gentlemen whose names appeared on the balloting list, inasmuch as he had never yet been formally presented to the Institute and gone through the initial proceeding on the reception of a member as required by the rules and regulations. Officially, as President of the Actuarial Society of America, he thought he could say without derogation to the dignity of that body, or without in any way being objectionable to any member, that as an organization they felt they were as yet inferior to the Institute of Actuaries. The Institute had its age, nearly 50 years of valuable work to look back to. Its members had their professional standing secured, partly by the great work which had been accomplished, partly by the Royal Charter; so that their position was more respected in England and the British Empire generally, than the position of an actuary in the United States had yet become. Then, again, the Institute had this venerable hall to meet in, with all the associations of the past few years, and with many venerable associations which must make it more interesting. On their part, they had no place of meeting, except such as they might happen to secure for the time being. He would not go on enumerating the different points, but from almost every point of view the Institute possessed the advantage. Still speaking of the body of which he was president, it was organized only six years since, and had not as yet devised a system by which persons could be admitted on their merits by examination. Their rule at present was to admit no one until he

had shown by some published work his competence for admission. This test was not embodied in the bye-laws, but it was understood amongst the members of the Council. The position of an actuary in the United States differed in two respects certainly from that of an actuary in this country. An actuary in the United States, as such, was not the principal officer of his company, although the principal officer in some cases was an actuary and a member of their society. This difference was not so important as it might seem, because every actuary was able to make felt his decisions in his own company if he was capable of doing so, and he did not think that actuaries felt any loss of respect, or genuine position, by not being, in many cases, as they were here, the managing officers of their companies. On the other hand, the position of an actuary in the United States in his judgment—and he spoke for no others—had been damaged by the system of official valuations. The different States made yearly official valuations of the various companies on a uniform standard. The result of that system had been what probably no one would have expected. The officer of the State whose business it was to make these annual valuations went to a published set of tables, put down the figures which he found there, added them up, and that was the valuation. The actuary might, and usually did, make his own valuation for his own company, very often on a different standard, but in the eye of the public the valuation was made by a state official and not by the actuary of the company. He would leave it to those present to judge what the position of actuaries in this country would be if they were not held responsible for their own valuations. That particular element of professional authority which came from making the valuation was largely lost in his country, for the simple reason that a person was accustomed to refer to the superintendents of the insurance department of his own State, or some other State, to find out what the total reserve liability was. These statements were published widely, and were in the hands of a great many agents and a large number of people who were not agents, and they knew as much, or thought they did, about the value of a policy as any actuary. The public seemed to think that they must have as the surrender-value the amount of reserve on their policies, and they had to a certain extent an excuse owing to the system of Government valuations. He took the greatest pleasure in entering for the first time on the duties and privileges as a member of the Institute.

On the motion of Mr. W. P. Phelps, seconded by Mr. G. J. Lidstone, a vote of thanks was given to Messrs. Miller, Moore and Holt, for their services as Auditors during the past year.

The President then announced that the meeting was adjourned to 25 November next.

COLONIAL EXAMINATIONS.

Examinations were held on 19 and 20 April 1895, at Sydney, Melbourne, Wellington, Montreal, and Toronto, with the following results:

PART I.

Thirty Candidates sent in their names, of whom fourteen presented themselves, and ten passed as follows:

Class I:
Griffin, A. D. (Toronto).
Haight, Milton (Toronto).
Treleaven, W. (Sydney).
Wickens, C. H. (Melbourne).

Class II:
Doyle, A. J. (Sydney).
Neighbour, G. H. (Melbourne).

Class III:
Anderson, A. T. (Sydney).
Emery, J. M. (Toronto).
Kingsbury, J. W. (Sydney).
Ross, C. W. (Melbourne).

PART II.

Seventeen Candidates sent in their names, of whom thirteen presented themselves, and five passed as follows:

Class I:
None.

Class II:
Bradshaw, T. (Toronto).
Galwey, C. S. (Wellington).
Thodey, R. (Melbourne).

Class III:
McMinn, W. J. R. (Montreal).
Paradice, W. H. (Sydney).

PART III (SECTION A).

Of the two gentlemen who sent in their names, one presented himself and passed, namely:

Class III:—Day, W. R. (Sydney).

PART III (SECTION B).

One gentleman sent in his name for this section of the Final Examination, and passed, namely:

Class III:—Muter, Percy (Wellington).

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On a method of Computing the Temporary Deductions to be made from the Sums Assured, upon Rated-up Lives, in lieu of Extra Premium. *By* GEORGE F. HARDY, F.I.A., *Actuary of the English and Scottish Law Life Assurance Association.*

[Read before the Institute, 29 April 1895.]

THE general subject of extra risks, and the special method of meeting such risks by making temporary deductions from the sum assured in lieu of charging an extra premium, have been more than once under discussion at this Institute. The latter problem in particular was somewhat exhaustively considered upon the occasion of a paper by the late Mr. Sunderland, "On a Method frequently adopted of treating Under-Average Lives for Assurance purposes, &c." (*J.I.A.*, xxix, 419).

I do not propose here to re-open the general question, but having lately had occasion to compute a table of annually diminishing deductions equivalent to various degrees of rating-up, a short account of the method employed, which I believe to be novel and to have some advantages to recommend it in practice, may be of interest to the members of the Institute.

The method is based upon two assumptions:

1. That the extra mortality is confined to the period during which the policy is subject to a deduction (in practice the "expectation of life", $\mathring{e}_x$), and that thereafter the policy may be treated for all purposes as having been effected at the ordinary rate of premium.
2. That the extra risk is so distributed throughout the above-mentioned period that the reserve value of a policy subject to such temporary deduction is the same as that of an ordinary policy of the same age at entry and duration.

The convenience of both these assumptions will be sufficiently obvious, and it will probably be admitted that on the whole they represent the facts, so far as they are known, quite as well as the assumption that the extra mortality is uniform throughout life, and better than the assumption that such extra mortality can be represented throughout life by a constant addition to the age.

As a consequence of the above assumptions we have the following equation connecting the rates of mortality on normal and rated-up lives, for entrants at age x, t years after entry

$$q'_{x+t}[1-(m-t)\mathrm{X}-{}_{t+1}\mathrm{V}_x] = q_{x+t}(1-{}_{t+1}\mathrm{V}_x) \quad . \quad . \quad \text{(i)}$$

where m = the term during which the policy is subject to the annually diminishing deduction, and $m\mathrm{X}$ = the initial value of such deduction.

The equation (1) follows of course from the consideration that, starting the tth year with the same reserve value for both the ordinary and rated-up assurance, receiving the same annual premium, and closing the year again with equal reserve values in each case, the expected death-strain (that is, the product of the expected rate of mortality by the sum assured less the reserve held against the policy) must also be the same in each case.

If we assume that the policy carries a reversionary bonus B, which is subject to a proportionate deduction with the sum assured, equation (1) will be modified as follows:

$$q'_{x+t}\{[1-(m-t)\mathrm{X}](1+\mathrm{B})-{}_{t+1}\mathrm{V}_x-\mathrm{B}\,\mathrm{A}_{x+t}\} = q_{x+t}\{1+\mathrm{B} \\ -{}_{t+1}\mathrm{V}_x-\mathrm{B}\,\mathrm{A}_{x+t}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

It is clear that X and q' are each implicit functions of the other; as however q'_{x+t} is also a function of t, while X is constant with respect to t, it is evidently necessary to invert the usual process of treating the mortality as the known, and the amount of debt as the unknown quantity, and to determine the mortality rates corresponding to various values of X. In other words, to find the amount of rating-up corresponding to given values of X.

Having given for example the value of X and m for a given age at entry (x), assuming a given mortality table and rate of interest, we can compute the values of q'_{x+t} for all values of t (up to $t=m$) from equations (i) or (ii) as the case may be. From these values the corresponding annuity-value a'_x may be deduced, and the equation

$$a'_x = a_{x+r}$$

will give at once the number of years of rating-up (r), equivalent to the assumed value of X.

The entire process would of course be extremely laborious, but by an extensive use of interpolation, and of simple summation formulas, the whole work is reduced to a comparatively small compass.

The process will be best understood by the aid of the tables appended, which show practically the whole of the working. The H^{M} Table (*Text-Book* Graduation) at 3 per-cent interest was employed; and, taking the three ages at entry, 20, 40 and 60, the rating-up equivalent to initial deductions (diminishing annually) of 60 per-cent and 30 per-cent respectively of the sums assured and bonus additions, was computed in each case as shown in Table I. From these results, merely by a series of interpolations, the figures in Tables II, III and IV were derived.

In order to simplify the work, equation (ii) may be modified, by using the forces of mortality μ'_{x+t} and μ_{x+t} in lieu of q'_{x+t} and q_{x+t}, the values of ${}_{t-1}V_x$ and A_{x+t} being replaced by the suitable continuous functions (${}_tV_x$, the reserve value at the moment of time t, and A_{x+t}).* Finally, since

$$-\mu_x = \log_e p_{x-\frac{1}{2}} \text{ nearly,}$$

* Strictly, the use of μ_{x+t} in lieu of q_{x+t} involves the substitution of ${}_tV_x + \frac{1}{2}P_x$ for ${}_{t-1}V_x$, and of $(m-t-\frac{1}{2})X$ for $(m-t)X$ in the formulas (i) and (ii). It was considered sufficiently accurate for the purpose in view, however, to use the values of ${}_tV_x$ and $(m-t)X$, leading to equations (iii) and (iv).

the latter function was substituted for μ and, assuming B to represent a simple reversionary bonus of £2 per-cent per annum, equation (ii) took the form (omitting suffixes for the sake of clearness)

$$\log p'\{[1-(m-t)X](1+\cdot 02t)-V-\cdot 02tA\}$$
$$=\log p\{1+\cdot 02t-V-\cdot 02tA\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

whence
$$\frac{\log p'}{\log p}=\frac{1+\cdot 02t-V-\cdot 02tA}{[1-(m-t)X](1+\cdot 02t)-V-\cdot 02tA}$$
$$=R_t \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The values of this ratio (R_t) for each age at entry, and for values of t equal to $\frac{m}{4}$, $\frac{m}{2}$, $\frac{3m}{4}$, and m are given in the third column of Table I, when $mX=\cdot 60$ and in the tenth column, for $mX=\cdot 30$. From these values the next columns ($\delta \log p=\log p' - \log p$) are easily derived from the simple relation

$$\frac{\log p'}{\log p}=R_t, \text{ as above}$$

$$\therefore \qquad \log p'-\log p=(R_t-1) \log p.$$

The columns headed $\log D'-\log D$ show the amount by which the function $\log D_{x+t}$ is affected by the increased values of $\log p'$ for all ages between $x+t$ and $x+m$, the values of D'_{x+m} and D_{x+m} being taken as identical (as in the case of all functions for ages above $x+m$).

The values of $\log D'-\log D$ in columns (5) and (12) were obtained from the values of $\delta \log p$, in the preceding columns by the following approximate formulas of summation:

$$\int_{3t}^{4t} u_x dt=\frac{t}{24}\left(9u_{4t}+19u_{3t}-5u_{2t}+u_t\right) \quad . \quad . \quad . \quad \text{(v)}$$

and
$$\int_{2t}^{4t} u_x dt=\frac{t}{3}\left(u_{5t}+4u_{3t}+u_{2t}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

by writing $\frac{n}{4}$ for t and $\delta \log p$ for u_x.

The latter of these is a well-known formula, and the former may easily be demonstrated as accurate to third differences.

From the values of $\log D'-\log D$ it is, of course, easy to pass to the values of $D'-D$ (columns 7 and 13) and thence by use of

formula (vi) to the value of $\overline{\mathrm{N}}'-\overline{\mathrm{N}}$. Having now obtained the values of N'_x and D'_x, we have at once the resulting value of $\bar{a}'_x$, and from this, finally, we can determine the amount of rating up (r) from the equation $a'_x=a_{x+r}$, which corresponds to an initial deduction of 30 or 60 per-cent of the sum assured, as the case may be, at age at entry x.

These results are given for the six cases in which they were computed in Table I, and it will be noticed that the function $a-a'$ changes very slowly with the age at entry, and is therefore of the greatest assistance in the interpolation of the intermediate values of Table II.

It will also be noticed that the values of the quantities $a-a'$ are more nearly proportional to the values of $\dfrac{m\mathrm{X}}{1\cdot2-m\mathrm{X}}$ than to $m\mathrm{X}$.

Having obtained the values of $a-a'$ and the corresponding values of r given in Table II for decennial ages at entry and intervals of ·10 in the value of $m\mathrm{X}$, the values of r for smaller intervals as given in Table III were readily interpolated, and, in fact, might be written down by inspection.

Finally, by entering Table III inversely, the values of $m\mathrm{X}$, the amount of deduction from each £1 assured in the first year of assurance, corresponding to a rating-up of 3, 5, 7 or 10 years, were obtained and are given in Table IV.

A comparison of these values of $m\mathrm{X}$ with those given in Mr. Sunderland's paper shows a fair agreement between the two sets. In making the comparison, however, it must be borne in mind that the values of $m\mathrm{X}$ here given would have been somewhat diminished had a 4 per-cent rate of interest been used in the annuity-values, and on the other hand they would have been increased by omitting the reversionary bonus from account.

TABLE I.

Age at Entry $=(x)$	t	$mX=\cdot60$: $\frac{\log p'_{x+t-\frac{1}{2}}}{\log p_{x+t+\frac{1}{2}}}$	$\delta \log p = \log p' - \log p$	$\log D' - \log D = \int_n^t \delta \log p \,.\, dt$	D_{x+t}	$\delta D_{x+t} = (D'-D)$	$\delta N_{x+t} = \bar{N}' - \bar{N}$	Equivalent Surcharge in Years	$mX=\cdot30$: $\frac{\log p'_{x+t-\frac{1}{2}}}{\log p_{x+t-\frac{1}{2}}}$	$\delta \log p = \log p' - \log p$	$\log D' - \log D = \int_n^t \delta \log p \,.\, dt$	$\delta D_{x+t} = (D'-D)$	$\delta N_{x+t} = \bar{N}' - \bar{N}$	Equivalent Surcharge in Years
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
	0	2·50	·0037	·115	53,188	16,200	177,100	12·3	1·43	·0011	·039	5,000	60,200	1·4
20	$10\frac{1}{2}$	2·06	·0036	·076	36,273	6,900	whence	...	1·34	·0011	·028	2,400	whence	...
$(m=\cdot42)$	21	1·66	·0030	·041	21,213	2,100	$\bar{a}'_x=19\cdot85$	...	1·24	·0011	·016	900	$\bar{a}'_x=21\cdot66$	...
	$31\frac{1}{2}$	1·30	·0022	·014	15,508	500	$a-a'=2\cdot71$	...	1·13	·0009	·006	200	$a-a'=\cdot90$	...
	42	1·00	·0000	·000	...	...		...	1·00	·0000	·000	...		...
	0	2·50	·0066	·132	25,223	8,950	70,090	8·0	1·43	·0019	·046	2,820	24,260	3·0
40	$6\frac{3}{4}$	...	·0062	·089	19,114	4,350	whence	...	...	·0020	·033	1,510	whence	...
$(m=27)$	$13\frac{1}{2}$	...	·0056	·049	14,107	1,680	$\bar{a}'_x=15\cdot10$	...	...	·0020	·020	660	$\bar{a}'_x=16\cdot76$	...
	$20\frac{1}{4}$	...	·0010	·016	9,552	360	$a-a'=2\cdot58$	...	...	·0017	·007	150	$a-a'=\cdot92$	...
	27	1·00	·0000	·000	...	...		...	1·00	·0000	·000	...		...
	0	2·50	·0197	·184	9,988	5,270	20,750	6·5	1·43	·0056	·065	1,610	7,175	2·4
60	$3\frac{1}{2}$	...	·0181	·118	8,017	2,500	whence	...	...	·0058	·044	860	whence	...
$(m=14)$	7	...	·0149	·059	6,207	900	$\bar{a}'_x=8\cdot38$	...	...	·0054	·024	370	$\bar{a}'_x=9\cdot85$	...
	$10\frac{1}{2}$	...	·0090	·017	4,577	180	$a-a'=2\cdot34$	...	...	·0039	·008	90	$a-a'=\cdot87$	...
	14	1·00	·0000	·000	...	...		...	1·00	·0000	·000	...		...

TABLE II.

Differences between the normal and rated-up annuity-values $(a-a')$ and equivalent rating-up in years (r) corresponding to given values of mX.

Age at Entry	$m\text{X}=\cdot60$		$m\text{X}=\cdot50$		$m\text{X}=\cdot40$		$m\text{X}=\cdot30$		$m\text{X}=\cdot20$		$m\text{X}=\cdot10$	
	$\frac{m\text{X}}{1\cdot2-m\text{X}}=1\cdot00$		$\frac{m\text{X}}{1\cdot2-m\text{X}}=\cdot71$		$\frac{m\text{X}}{1\cdot2-m\text{X}}=\cdot50$		$\frac{m\text{X}}{1\cdot2-m\text{X}}=\cdot33$		$\frac{m\text{X}}{1\cdot2-m\text{X}}=\cdot20$		$\frac{m\text{X}}{1\cdot2-m\text{X}}=\cdot09$	
	$a-a'$	r	$a-a'$	r	$a-a'$	r	$a-a'$	r	$a-a'$	r	$a-a'$	r
20	2·71	12·2	1·91	8·9	1·33	6·3	·90	4·3	·55	2·7	·26	1·3
30	2·66	9·8	1·90	7·2	1·35	5·2	·92	3·6	·57	2·3	·27	1·1
40	2·58	8·0	1·86	5·8	1·34	4·2	·92	3·0	·57	1·9	·27	0·9
50	2·47	6·8	1·80	5·0	1·30	3·6	·90	2·5	·56	1·6	·27	0·8
60	2·34	6·4	1·72	4·7	1·25	3·4	·87	2·4	·55	1·5	·26	0·7

TABLE III.

Values of (r) for given values of mX (obtained by interpolation from Table II).

Age at Entry	VALUES OF r WHEN $m\text{X}=$ ·60	·55	·50	·45	·40	·35	·30	·25	·20
20	12·2	10·5	8·9	7·5	6·3	5·2	4·3	3·5	2·7
30	9·8	8·4	7·2	6·2	5·2	4·4	3·6	2·9	2·3
40	8·0	6·8	5·8	5·0	4·2	3·6	3·0	2·4	1·9
50	6·8	5·8	5·0	4·3	3·6	3·0	2·5	2·0	1·6
60	6·4	5·5	4·7	4·0	3·4	2·9	2·4	1·9	1·5

TABLE IV.

Values of m and mX corresponding to a rating-up of 3, 5, 7 or 10 years.

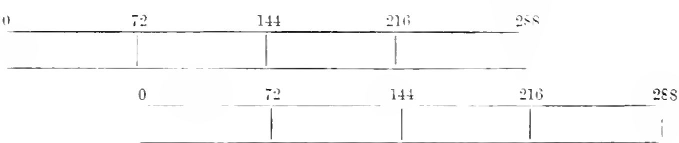
Age at Entry	m	mX Rating-up 3 Years	Rating-up 5 Years	Rating-up 7 Years	Rating-up 10 Years
20	42	·22	·34	·43	·54
25	38	·24	·36	·46	·57
30	35	·26	·39	·49	·61
35	31	·28	·42	·53	·64
40	27	·30	·45	·56	·67
45	24	·33	·48	·59	...
50	20	·35	·50	·61	...
55	17	·36	·51	·62	...
60	14	·36	·52	·62	...

On a New Method of performing approximately certain Operations in Multiplication and Division. *By* J. A. Robertson, A.I.A., A.F.A., *of the Guardian Fire and Life Assurance Company, Limited.*

[Read before the Institute, 29 April 1895.]

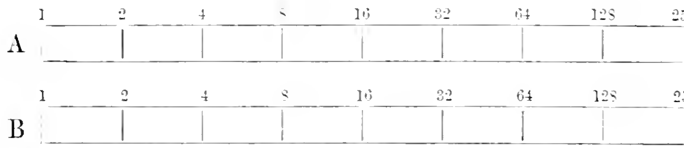
1. The method about to be described is based upon a special application of the principle involved in the slide-rule. As this instrument is less familiar now than it was formerly, and as its working presents many points of interest, I propose to begin with a short account of its construction and use.

2. An elementary form of slide-rule calculation may be illustrated by using two common scales to perform additions. With two ordinary yard-measures subdivided into, say, eighths of an inch, and having these divisions numbered consecutively from 1 up to 288, the addition of any two numbers whose sum does not exceed 288, may be performed by a purely mechanical operation. Suppose the numbers to be 72 and 144. The two scales being placed side by side in the same horizontal plane, the edge of the one free to slide along the edge of the other, you draw one of them—say the lower—towards the right hand, until its extremity on the left hand lies immediately below one of the given numbers—say 72—on the upper scale. The distance at which the other number, 144, on the *lower* scale now stands from the left-hand extremity of the upper scale is 72 + 144 (= 216) divisions, and 216, the sum required, will be the number on the upper scale immediately above 144 on the lower, as in this figure:



The rule for procedure would be stated thus: place the left-hand extremity of the lower scale under one of the two numbers on the upper scale, and find the sum required on the upper scale immediately above the other number on the lower scale. As a corollary it should be noted that when the left-hand extremity of the lower scale has been placed below any number, m, on the upper scale, you will find on the upper scale above the series of numbers n_1, n_2, n_3, &c., on the lower scale (so long as the summations are within the limits of the scale) the values of $m+n_1$, $m+n_2$, $m+n_3$, &c.

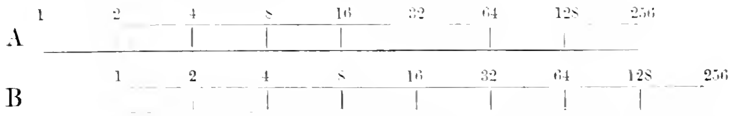
3. The mode of using the instrument specifically called the slide-rule, follows precisely the same lines, but as the two scales of which it consists are logarithmic, the result of the operation is the *product,* not the *sum* of the two numbers being dealt with. In the common scale the distances from the left-hand extremity to the points marked 1, 2, 3, &c., are proportional to these numbers; in the logarithmic scale the distances are proportional to the logarithms of the numbers. The following figure will help to explain the construction of the instrument:



It will be observed that on these scales the equal divisions, which on the common scale would be marked 0, 1, 2 . . . 8, are marked 1, 2, 4 . . . 256; in other words, they are marked 2^0, 2^1, 2^2 . . . 2^8, and we know from the definition of a logarithm that 0, 1, 2 . . . 8 are the logarithms, to base 2, of the numbers 1, 2, 4 . . . 256. As they stand, these scales give, in their relative positions, certain selected numbers from a complete slide-rule, and they enable us to multiply together, by a purely mechanical operation, any two of these numbers whose product does not exceed 256. Thus, to multiply 16 by 2 you draw Scale B to the right hand until its left-hand extremity lies immediately below one of the factors—say 2—on Scale A, and then above the other factor, 16, on B, you find on A the required product, 32.* The rationale of the operation is sufficiently obvious: by sliding Scale B to the right until its left-hand extremity coincides with 2 on Scale A, the distance at which 16 on Scale B now stands from the left-hand extremity of Scale A has been increased by a space corresponding to the logarithm of 2, which distance is represented on Scale A by the space corresponding to the sum of the logarithms of 16 and 2—that is, to the logarithm of 32. As in the case of the two common scales, each adjustment of the lower scale gives a number of results, and this is an important feature in slide-rule working. When the left-hand extremity of Scale B has been placed below any number, *m,* on

* The best way to familiarize one's self with the working of these scales, and the others to be described, is to cut a piece of cross-ruled paper into strips, number them appropriately, and then move them about into various positions.

Scale A, you will find on Scale A above the series of numbers n_1, n_2, n_3, &c., on Scale B (so long as the products are within the limit of the scale) the values of mn_1, mn_2, mn_3, &c. The following figure shows the scales placed in position for the multiplication of 16 by 2, and exhibits the various other products by 2 which this one setting of the scales furnishes:



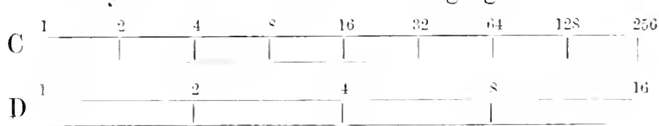
The general relation between the two scales may be expressed in graphic form thus:

$$\begin{array}{cccc} \text{A} & m & \dfrac{mk}{n} & k \\ \hline \text{B} & n & k & \dfrac{nk}{m} \end{array}$$

to be read in this way: Set m on A to n on B; then over k on B find $\dfrac{mk}{n}$ on A, or under k on A find $\dfrac{nk}{m}$ on B. In other words, when any two numbers, m on A and n on B, are placed opposite each other, then of all the other pairs of numbers similarly situated, the number on A is to the number on B as m is to n. The rule already given for multiplication is now seen to be simply a particular case of this general expression—that case, namely, in which n is taken equal to unity.

4. The principle of the slide-rule is capable of considerable extension, and when two different scales are used in combination, results of an interesting kind may be obtained. A brief mention of two of these variations will serve to indicate the scope and power of this method of calculating.

(1) On the ordinary slide-rule there is generally given, in addition to the two logarithmic scales whose employment has just been described, another pair of scales called the C and D lines. C is a logarithmic scale precisely similar to A and B; D is also a logarithmic scale, but on it the distance at which each number stands from the left-hand extremity of the scale is twice as great as the distance at which the same number stands from the left-hand extremity of Scale C—see the following figure:

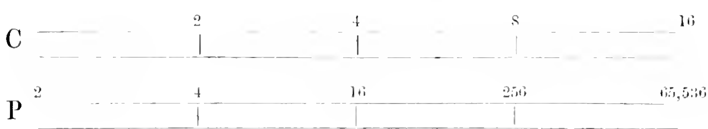


The general relation between the two scales, and the results obtainable by their combined use, may be represented in a form analogous to that used in Art. 3:

C	m	$\frac{mk^2}{n^2}$	k
D	n	k	$n\sqrt{\frac{k}{m}}$

A variety of special relations might be indicated by taking m or $n=1$, or $m=n$; but, without going into further details (which are not essential to our main purpose at present), it may be sufficient to say that these scales afford facilities for dealing with mean proportionals, areas of circles, mensuration of solids, and the like. Questions relating to timber-measurement and cask-gauging (the class of work to which the practical application of the slide-rule is chiefly confined) generally require for their solution the C and D lines in combination with certain constant quantities such as the ratio of the area of a circle to its diameter, the relations between the different "standards" according to which timber is sold in different countries, and so on. These constant quantities are called "gauge-points", and a selection of those most frequently occurring in practice is sometimes tabulated on the back of the rule.

(2) Another development of the principle is seen in a form of slide-rule recently patented by Captain Thomson, of the Woolwich Arsenal. Of the two additional scales, C and P, which he uses in combination, C is the ordinary logarithmic scale, and P is a special scale (he calls it a *Power* Scale) on which the distance between any number marked thereon and the left-hand extremity is proportional to the logarithm of the logarithm of the number. In their simplest form these scales would appear thus:



and the general relation between the two scales might be expressed in this way:

C	m	$\frac{m \log k}{\log n}$	k
P	n	k	$n^{\frac{k}{m}}$

A little consideration will show that these scales afford the means of performing operations into which involution and evolution enter, a single setting giving a series of powers of the same factor. Logarithms to a given base can be obtained by setting 1 on C to the base on P, and then opposite any number on P will be found on C its logarithm to that base. If 1 on C be set to $(1+i)$ on P (i being the effective rate of interest), then opposite any number of years on C there will be found on P the accumulated amount of 1 for that period, and the accumulation of any given amount can then be found by using the ordinary A and B scales for the necessary multiplication.

5. In theory the slide rule may be regarded as a perfect instrument, alike as regards accuracy of construction and simplicity and rapidity of working; but its employment in actual practice is found to be attended by a drawback of a serious kind.

Before looking at this, there is a preliminary point to be considered. If reference be made to Art. 3, fig. 2, it will be seen that the product of 256 by 2 is not exhibited on account of Scale B having overshot Scale A. The usual way of providing for this is to duplicate each scale, so that its natural termination at 100, 1,000, or 10,000, appears in the middle, and is followed by a precisely similar subdivision. Another method of getting over the difficulty, at once simpler and more elegant, is to present the slide-rule in a circular form, Scale B being set off on the outer edge of a disc placed so as to revolve within a fixed circle, whose circumference is similarly divided and corresponds to Scale A. In this way an endless scale is provided, but with this disadvantage, that it is not possible to adjust it so accurately as the ordinary straight scale.* I have been informed by Captain Thomson that he has had constructed for his own use a calculating disc embodying the principle of his power-scale (see Art. 4 (2)), and that he has employed this largely in actual work.

The drawback to the practical application of the slide-rule is

* A small scale in this form—so small as to be only a toy—has recently been published under the name of "Saxon & Co.'s Automatic Calculator." The author of the directions furnished for its use has fallen into some extraordinary errors in describing the instrument, with whose properties, indeed, he seems to have some acquaintance, but neither with its name nor its underlying principle. He says, "It will be observed that the space between 1 and 2 is larger than any of "the others and that there are more lines subdividing that interval [which is "true enough]. The intervals between 2 *to* [*sic*] 3, 3 to 4, are alike [which is "obviously incorrect] and larger than the following ones, *which do not differ in* "*size from each other*." (!) One would have thought that his eye would have shown him the incorrectness of these statements, even although his knowledge was insufficient to reveal their inherent absurdity.

occasioned by the fact that in order to give even four figures in its products, the instrument in its ordinary form would need to have so many subdivisions that its size would become altogether unmanageable, or its results be practically unreadable. In the preface to his *Logarithms and Anti-logarithms*, Mr. Erskine Scott mentions a form of slide-rule which is furnished with "a simple microscope" to facilitate its being accurately set and read, but it is obvious that work carried on under such conditions would involve a very serious strain on the eyesight of the operator. Various devices have been adopted with the view of bringing a large scale within convenient limits, and some of these are described by the late General Hannyngton in an interesting note to Part I of the *Institute of Actuaries' Text-Book*. He mentions there his own form of extended slide-rule (a most ingenious duplication of the scales, which is fully described and clearly illustrated in a paper by Mr. J. A. Galbraith, M.A., printed at the Dublin University Press); Professor George Fuller's arrangement, in which a scale of some 40 feet in length is wound in a spiral round a revolving cylinder capable of being held in the hand, and which is used in some insurance offices; and a form invented by the late Mr. Thomas Dixon, of Buttershaw, in which the spiral curve is described on a flat surface. To these may be added the Patent Calculator of M. Boucher, of Paris, in shape and size like an ordinary watch, in which the scale is laid down in four concentric circles. In the three last cases the setting and reading are done by means of indices variously arranged. General Hannyngton points out that of these extended forms of the instrument his own is the only one in which there is preserved that important property of the slide-rule—its power of giving, at each setting, as many results as may be desired in the same proportion; and he says, "A seemingly unavoidable defect of spiral arrangements is "the necessity for moveable indices that require special adjust-"ment and give but one result at one time."

6. This "seemingly unavoidable defect" is capable of explanation, and its investigation has a direct and important bearing on the method about to be set forth. To understand it we must go back to the original process of which the slide-rule, as we know it, is a development. In 1624, some 10 years after Napier had made known to the world the theory of logarithms, Gunter invented what he called the "line of numbers", and which was simply a logarithmic scale. In order to find thereon the product

of two numbers—say m and n—he used a pair of compasses: extending these from the left-hand extremity of the scale to the point marked by one of the factors—say m—he applied the same extent of the compasses to the other factor, n, and thus reached the required product. In other words, he found his product as far distant from the one factor as the other factor stood from the left-hand extremity of the scale. Working in this way he would, of course, only get one multiplication by m at a time; each fresh product—say mn'—would require the compasses, still remaining open to the same extent, to be applied to n'. It was not until the *second* scale was introduced, and Gunter's line of numbers was modified into the slide-rule, that it became possible to exhibit at one view a series of values in the same proportion; and the reason for the "defect" in the spiral arrangements just described is that they are single scales only, reversions to the original type. It is not worth while going into the matter here, but if anyone will examine carefully the directions given for setting the indices on, *e.g.*, Fuller's scale, he will find that the process corresponds exactly with the application of the compasses to Gunter's line of numbers.

7. We have now reached the point at which the discovery of the new method of calculation occurred, and may proceed to its consideration. Stated briefly, the process consists in this—a special application to a table of anti-logarithms of the principle followed by Gunter in working with his line of numbers.

Appended to this paper (see page 180) there will be found a table of anti-logarithms (which it is proposed to publish separately in card form)—a re-arrangement of the ordinary four-figure table, with the proportional parts omitted.* This may be looked upon as a continuous scale, divided into 1,000 equal spaces, and having certain numbers, ranging from 1000 to 9977, distributed over these spaces in such a way that the distance at which each number stands from the beginning of the scale is proportional to its logarithm, these distances being indicated by the numbers in the margin combined with those at the tops of the columns. Thus, the number 3155 is seen to occupy the 499th space, and 499 is the logarithm to three places of 3155. The difference between this scale and Gunter's line of numbers lies simply in this: in the latter the numbers are first selected at regular intervals, and then

* The setting out of the table in a single column of ten subdivisions, instead of the usual double column, was suggested to me by my friend Mr. James Chatham, and possesses many advantages.

placed at the points corresponding to their logarithms: the former is first subdivided into a number of equal spaces, and then in each one of these is placed the appropriate anti-logarithm. The marking of Gunter's line corresponds with that of the slide-rule in its ordinary form: the marking of the anti-logarithm table corresponds with that of the slide-rule in the illustration in Art. 3, with this difference, that at the equidistant points are placed powers of 10 instead of powers of 2. As in Gunter's line, so in the anti-logarithm table, the distance of a product from the beginning of the scale is equal to the combined distances of the factors, and, conversely, the distance of a quotient is equal to the difference between the distances of the dividend and of the divisor.

8. A simple example will serve to illustrate the application of the method to multiplication. Suppose the product required be 281 × ·417 (= 117·177).

(1). To do this on Gunter's line of numbers you would require a scale divided into 1,000 parts and then repeated as described in Art. 5. You would measure with the compasses the distance from the beginning of the scale to 281, and then apply this distance to 417 which would bring you up to a point very slightly beyond 117 on the second part of the scale, and set down your result as 117.

(2). On the anti-logarithm table you would look out the nearest number to 281, which you find to be 2812, occupying the 449th space from the beginning of the scale. You would then place one finger of the left hand on 4169 (the nearest number to 417) and advance 449 spaces from that point, keeping in view that when you get to the end of the table you simply go back to the beginning and continue counting, as if it were an endless scale. This brings you up to the space marked with the number 1172, and you set down your result as 117·2.

9. As regards this advance of 449 spaces a little further explanation seems desirable. In making the advance, advantage is taken of the symmetrical arrangement of the table. Each line contains 10 spaces, each group of 10 lines, clearly marked off by the ruling, contains 100 spaces. Having as your starting point the space occupied by 4169, you place the pen or pencil at once on the corresponding space four *groups* in advance, which is occupied by 1047: you then pass down four *lines* to the space occupied by 1148; and finally you advance nine *spaces*, and reach the space occupied by 1172. The operation is one of simple inspection, and

the rule of procedure may be thus stated (it being understood that in all cases logarithms to three places are being dealt with): Count on from the space occupied by the multiplicand as many *groups* as there are units in the first figure of the logarithm of the multiplier, as many *lines* as there are units in the second figure, as many *spaces* as there are units in the third. The process might of course be reversed: in the above example you might first advance nine *spaces* to 4256, then four *lines* to 4667, and finally four *groups* to 1172; but the other method is not only easier—the steps being taken in the order in which the digits in the number 449 are mentally pronounced—but more scientific as well, being in accord with the principle of performing numerical operations, as far as possible, from left to right.

In describing the process thus minutely, with a view to perfect clearness, it has been made to appear somewhat more complicated than it really is; practice will soon give the requisite dexterity, and the operation will be found extremely simple. The general rule stated above may occasionally be modified according to the taste of the individual computer. In practically employing the method myself I have found that the easiest way to take the first step (the one involving the *groups*) is to make use of the figures in the margin. Thus, starting with 4169, I note that it is in the line marked 62 and that four groups in advance will bring me to the corresponding space in the line marked 2, and I place my pen there at once. Then the simplest way to advance the remaining 49 spaces is to go on five lines (*i.e.* 50 spaces) and go back one space. Sometimes it will be found easier to go backwards to the extent of the arithmetical complement: if, *e.g.*, the number of spaces to be advanced were 995 it would obviously be the best plan to go backwards to the extent of five spaces.

10. After what has been said it will not be necessary to go fully into the subject of division which is performed, as will be at once apparent, by simply reversing the process followed in multiplication: that is to say, by counting backwards instead of forwards. Thus, to divide 117·2, the product obtained above, by 281, one of its factors, you would, having found 2812 in the 449th space as before, go backwards (from 1172) 449 spaces; the halting points would be first 4667, then 4256, and finally 4169; the last, suitably pointed off, being the quotient required. As in the case of multiplication, so with division, it will sometimes be found more convenient to use the arithmetical complement and reverse the process.

11. It may be noted that the anti-logarithm table may be used to find approximately the reciprocal of a number. The number 2, *e.g.*, occupies the 301st space, the arithmetical complement of 301 is 699, and in the 699th space will be found the reciprocal, ·5000, the rationale of the process being self-evident. Otherwise stated, the reciprocal of a number will be found to occupy the space whose distance from the end of the table is one space less than the distance at which the number itself stands from the beginning of the table. I had sketched out for the table appended hereto a more symmetrical ruling, so that the marking would appear exactly the same if the table were turned upside down or supposed to be so turned. This would have enabled the operator, starting from a given space, to place his pen at once on the space equidistant from the end of the table, and so to find his reciprocal in the space following; but, as the results obtainable in this way are not very often required, I thought it a pity to alter the ruling with which computers have become so familiar, through its having been adopted both in Erskine Scott's Tables and the Institute Card. Besides, should a reciprocal be wanted at any time, it can be easily obtained by using the arithmetical complement of the position of the number as just explained.

12. It is now time to say something about the conditions under which this method of calculation may be employed in actual practice. Going back to the example in Art. 8 (2), and looking at the process adopted to ascertain the number of spaces to be advanced in respect of a given multiplier, we observe that it consists in using the anti-logarithm table to find the logarithm of the multiplier. It will be at once apparent that this mode of procedure, while a little cumbrous in itself, will not always give very good results (though, in occasional cases, it may be resorted to); and I would propose that, in general, the use of the method should be restricted as follows: to cases in which

(1.) The numbers dealt with are small, and the results need only be approximate; and

(2.) The logarithm of one of the factors is readily available.

As a further restriction, I would suggest that its employment should be, for the most part, limited to the checking of calculations, which have been already performed by some more exact process.

13. It will be noticed that the statement of these restrictions amounts virtually to a description of many of the calculations occurring in actuarial work. We are dealing every day with

functions which are numerically small, and whose logarithms have been tabulated; we shall often find it of advantage to have at hand a means of rapidly satisfying ourselves as to the approximate accuracy of results otherwise obtained; so that, in limiting the scope of the method in one direction, we have really opened up a wide field for its employment. For certain kinds of calculations occurring in life assurance offices, the method now under consideration will be found to afford exceptional facilities; and, by way of illustration, I have arranged some of these in two distinct groups. The logarithms mentioned are to be understood as taken to three places only.

14. The first group includes those cases in which a common factor, sometimes in combination with a constant addition, is involved.

(1) Half-yearly and quarterly premiums are frequently calculated by adding a percentage, say k_2 and k_4, to the yearly rate and dividing by 2 and by 4 respectively. If the logarithms of $\frac{1}{2}\left(1+\frac{k_2}{100}\right)$ and $\frac{1}{4}\left(1+\frac{k_4}{100}\right)$ be looked up once for all and recorded on the card itself, they will denote the number of spaces in advance of the yearly premiums at which the half-yearly and quarterly rates will be found.

(2) It may be desired to construct a table showing the sum which a monthly payment of, say, 5*s.* will assure at various ages. If these are to be derived by adding to the yearly premium for £100 a percentage k and a constant addition £C, the rule for obtaining the required amounts will stand thus: Add, mentally, to the office premium per £100, $\dfrac{\mathrm{C}}{1+\dfrac{k}{100}}$, advance the number of spaces equal to the logarithm of $\dfrac{1+\dfrac{k}{100}}{300}$, and then take the reciprocal in the manner described in Art. 11. A memorandum would be made on the card of the modified constant addition and the logarithm of the constant multiplier.

(3) On page xxix of the *Institute Tables* Mr. Peter Gray shows how to construct a table of the values of P_x with a loading of $17\frac{1}{2}$ per-cent. Starting from the same point as he does, the Commutation Table with the corresponding table of logarithms, and having noted on the card that the logarithm of 1·175 is ·070, you can, without any intermediate figures, write down the loaded

premiums (never more than one penny different from Mr. Gray's) thus:

	£	s.	d.
Age 10	1	5	0
„ 11	1	5	6
„ 12	1	6	3
„ 13	1	7	0

From M_x you advance the number of spaces equal to the co-logarithm of N_{x-1}, and then make a further advance of 70 spaces for the loading.

15. The second and more important group of calculations includes those in which, while the series of multipliers remains constant, each individual one varies according to age, years' duration, or the like. For work of this kind the arrangement of the table affords special advantages. The marginal column on the left hand (repeated on the right hand as well as on the extreme edges of the card), which in the ordinary anti-logarithm table represents the first two figures of the logarithm, and in our new way of looking at the table indicates successive lines of 10 spaces each, may be made to subserve sundry other purposes (see Table I, below).

We may first of all take the numbers which it contains, and which range from 0 to 100, to denote *ages*, and we may then insert opposite to them in the blank columns the logarithms of a variety of functions, each user of the card selecting those which occur most frequently in his own practice. The following will serve as an indication of what may be done in this direction.

(1) The logarithms of A_x, and of $A_{x.\overline{n}|}$ for the usual values of $x+n$, will often be found useful, as, *e.g.*, in valuing bonus additions when the reserve-value of a single policy is required; and the same logarithms used backwards may be employed in calculating reversionary bonuses corresponding to cash bonuses already ascertained. The use of the logarithms of $A_{x.\overline{n}|}$ may be still further extended. When logarithms to three places are being used it will be found that the logarithms of $A_{x.\overline{n}|}$, $A_{x+1.\overline{n}|}$, $A_{x+2.\overline{n}|}$, $A_{x+3.\overline{n}|}$ and $A_{x+4.\overline{n}|}$ may be taken, without any serious sacrifice of accuracy, as a series whose first differences are constant. That is to say—if we have tabulated the logarithms of $A_{x.\overline{n}|}$ when $x+n=60$, and when $x+n=65$, and then difference these logarithms for ages five years apart, we can get at the logarithms of $A_{x.\overline{n}|}$ when $x+n=61$, 62, 63, or 64; and if we set down not these differences themselves, but one-fifth part of them, we can obtain the required logarithm by inspection. A practical

illustration will make this clearer, and will exhibit the power of the new method as a labour-saving appliance. Suppose you have an endowment assurance policy, which, either through rating-up, or through some special fancy on the part of the policyholder, stands in the valuation sheets as payable at age 63 or death, the present age of the life assured being entered as 30; suppose, further, that you have allotted to this policy a cash bonus of £6·75, and that you wish to ascertain the corresponding reversionary bonus, on an H^{M} 3 per-cent basis.

(*a*) *By the Ordinary Process.*

From Hardy's *Valuation Tables* $_{\overline{32}|}a_{30} = 17{\cdot}598$.

From Orchard's or Rothery & Ryan's *Conversion Tables* $A_{30.\overline{33}|} = {\cdot}45831$.

From Crelle's *Tables of Products* $\dfrac{6{\cdot}75}{{\cdot}458} = 14{\cdot}7$, being the reversionary bonus required.

(*b*) *By the New Method.* (See Table I, below.)

Knowing that $\log A_{30.\overline{33}|}$ is, approximately, $= \log A_{27.\overline{33}|} + \frac{3}{5}(\log A_{32.\overline{33}|} - \log A_{27.\overline{33}|})$, we look in the third column opposite age 27, add, mentally, to the number shown, three times the number in the same line under the heading "$\frac{1}{5}\Delta$" which gives us 662; and then, placing one finger of the left hand on 6745 (the nearest number in the table to the cash bonus) we count backwards 662 spaces and get at once 14·7, the reversionary bonus required. Our card has thus enabled us to dispense not only with the necessity of referring to the three volumes mentioned above, but also with the necessity of multiplying columns in our valuation sheets.

(2) The logarithms of q_x will allow of a rapid check to be applied to a mortality investigation.

(3) The logarithms of the cash-values given for each £10 of reversionary bonus, and of the corresponding reductions of premium, will furnish a ready means of checking calculations involving these functions.

(4) Paid-up policies are sometimes calculated by dividing a large percentage—say k_1—of the reserve value by the single premium at the advanced age increased by a small percentage—say k_2—the formula being $\left({}_nV_x \times \dfrac{k_1}{100}\right) \div \left\{A_{x+n}\left(1 + \dfrac{k_2}{100}\right)\right\}$,

which may be written ${}_nV_x\left(A^{-1}{}_{x+n}\times\frac{k_1}{100+k_2}\right)$. Where this rule is adopted, the logarithms of this multiplier of ${}_nV_x$ should be recorded on the card opposite the successive ages (a very simple matter, involving merely the constant addition of the logarithm of $\frac{k_1}{100+k_2}$ to the co-logarithms of A_{x+n} which are probably already tabulated); and we can then, in whole-life cases, get the paid-up policy per-cent from the reserve-value by a single application of the card, and without the necessity of setting down any of the intermediate figures.

Hitherto we have been taking the marginal numbers to represent *ages*; we may also, if we choose, make them denote *years*.

(5) In this light a column might be filled up giving the values of $(s_{\overline{n+1}}-1)$—using the notation of King's *Theory of Finance*—at a given rate of interest. This would give a convenient means of calculating those accumulations of "premiums received" which are sometimes desired in connection with claims. Whatever may be thought of the propriety of using the new method in other cases, no one will dispute that here its results will be amply sufficient for practical purposes!

And once more, the marginal numbers might be taken to represent *days*. In dealing with ages and durations the table has been specially convenient, because we do not want to go beyond 100; in dealing with days we should require four columns in order to get up to 365.

(6) If the logarithms of each successive number of days when taken as decimals of a year (see *Jones on Annuities*, vol. i, pp. 71 and 72) were tabulated, calculations of proportional payments of premium might be checked with considerable ease. This arrangement is probably of less importance than some of the others already described, but it has been included as showing more fully the capabilities of the new method, and because reference is to be made to it hereafter.

The following table exhibits a partial filling up, in some of the directions just indicated, of a few of the blank columns with which the card is provided. It should be observed that the tabulating of the various logarithms, even when these are not given elsewhere and have simply to be copied out, is no formidable task. As they never exceed three figures, they may be written down very rapidly, and for the same reason they are easily

retained in the memory during the short space of time required to make a calculation.

Table I.

x	x = Age				x = Years	x = Days				x
	Log A_x H^M 3 %	Log $A_{x.n}$ $x+n=60$ H^M 3 %	$\frac{1}{3}\Delta$	Log q_x H^M	Log $(s_{\overline{x+1}}-1)$ 4 %	Log $\frac{x}{365}$,	$\frac{100+x}{365}$, &c. 100	200	300	
20	517	591	2·8	801	491	739	517	780	943	20
21	524	600	2·8	828	522	760	520	782	944	21
22	532	609	2·8	835	552	780	524	784	946	22
23	539	617	2·8	830	581	799	528	786	947	23
24	546	626	2·8	822	609	818	531	788	948	24
25	554	635	2·8	822	637	836	535	790	950	25
26	562	644	2·8	825	664	853	538	792	951	26
27	570	654	2·6	839	690	869	542	794	952	27
28	578	663	2·6	856	716	885	545	796	954	28
29	586	672	2·6	871	741	900	548	798	955	29
30	594	682	2·4	888	766	915	552	799	956	30

16. In describing the calculation of an endowment-assurance bonus [Art. 15 (1)] mention was made of Crelle's Tables. As these are largely resorted to in operations of the class we have been looking at, it may be not out of place to indicate how the results which they give compare with those obtainable by using the new method. (In relation to Crelle's Tables the *multiplier* will be understood to mean the number at the top of the column; in relation to the new method it will denote the number whose logarithm is given.)

When the factors to be multiplied together do not exceed three figures each, Crelle's Tables give the product exactly, and so long as one of the factors lies within this limit, the exact product can still be obtained with very little trouble, even though the other factor should extend to as many as six figures. But when both factors exceed the limit of three figures, it requires a somewhat cumbrous process to reach the exact product, and the use of the tables is chiefly restricted (so far as I am aware) to those cases in which the operator is satisfied with such results as he can get when he has cut down both his factors to three figures. (It should be noted that when he has cut down his multiplier he can no longer deal with a larger number of figures

in his multiplicand, as he could when his multiplier consisted of three figures exactly. His being able to rely in a general way on his results regarded as a whole, depends upon the individual variations over and under absolute accuracy balancing one another; if his multiplicands extend to four figures, and he gives effect to the fourth, his results will be always too great, or always too small, according as his multiplier, when cut down to three figures, is in excess or in defect.)

Limiting our examination to cases of multiplication of four figures by four figures, with results extending to four figures—*i.e.*, cases to which four-figure logarithms are applicable—will probably be considered sufficient for purposes of comparison. Here we have a range of 9,000 possible factors (from 1000 to 9999). The card adapted to the new method gives us 1,000 numbers within these limits from which to select the number nearest to the given factor, and from that factor we may advance by 1, 2, 3 . . . 1,000 different steps, getting a different product each time. Crelle's Tables give us only 900 different numbers from which to select (from 100 to 999), and the same number of products under each multiplier. If, with a given multiplicand, your multipliers range from 1005 to 1014, and you are using Crelle's Tables, you must take the column headed 101, and you only get one product; whereas with the card you may advance two, three, four, five or six spaces, and thus get five different products. Owing to the manner in which the numbers on the card are distributed this advantage necessarily diminishes as the first significant figure of the multiplier increases, but throughout more than three-quarters of its extent the card continues to give a greater selection than Crelle's Tables. In regard to the other factor the same thing holds good: when, with a given multiplier, your multiplicands range from 1005 to 1014, you can only deal with 101 in Crelle's Tables, whereas with the card you could begin making your advance from five different points—namely, 1005, 1007, 1009, 1012 and 1014. The general effect of this greater selection afforded by the anti-logarithm table is, of course, in favour of the new method; but cases will sometimes occur in which, although the anti-logarithm table may allow a closer approximation to be made in respect of each factor, Crelle's Tables may give the better product, on account of the greater discrepancies happening to be in opposite directions, and so counterbalancing one another.

The following table exhibits both processes in contrast in their application to the construction of a table of quarterly premiums when these are assumed to be derived from the yearly rates by adding 5 per-cent and dividing by 4. As the common factor, $\frac{1\cdot05}{4}$ $(=\cdot2625)$, is midway between two of the multipliers available in Crelle's Tables, I have shown the results obtained by multiplying both by ·262 and by ·263.

Table II.

Yearly Premium				Quarterly Premiums															
				By Crelle's Tables								By the New Method				By Actual Multiplication			
				Multiplier = ·262				Multiplier = ·263				419 spaces in advance				Multiplier = ·2625			
£	s.	d.	£ dec.	£ dec.	£	s.	d.	£ dec.	£	s.	d.	£ dec.	£	s.	d.	£ dec.	£	s.	d.
2	1	11	2·096	·5502	0	11	0	·5523	0	11	1	·5495	0	11	0	·5502	0	11	0
5	0	10	5·042	1·3205	1	6	5	1·3255	1	6	6	1·324	1	6	6	1·3235	1	6	6
6	2	8	6·133	1·6061	1	12	1	1·6122	1	12	3	1·611	1	12	3	1·6099	1	12	2
7	13	4	7·667	2·0095	2	0	2	2·0172	2	0	4	2·014	2	0	3	2·0126	2	0	3
9	7	1	9·354	2·4497	2	9	0	2·4591	2	9	2	2·455	2	9	1	2·4554	2	9	1
12	5	4	12·267	3·2236	3	4	6	3·2349	3	4	8	3·221	3	4	5	3·2201	3	4	5

It will be observed that the results (taken to the nearest penny throughout) given by the new method agree very closely indeed with those arrived at by actual multiplication (although—as anyone may see who will take the trouble to go through the calculations—no attempt has been made to select figures specially suitable for the anti-logarithm table), while Crelle's results show a good many deviations from absolute accuracy.

When we pass to the consideration of those cases described in Art. 15—cases in which the marginal numbers are brought into use—the advantages of the new method become much more apparent. Besides getting results which are better in the main, we have all our material collected on one card, instead of being distributed over at least two volumes—the Valuation Tables and Crelle's Tables; the card can be more conveniently laid upon the sheet we are working with than the two books, and its use involves no turning over of pages.

I should like it to be clearly understood that this comparison has been made solely with the view of justifying the introduction of the new method, and with no intention whatever of detracting from

the value of Crelle's Tables. There is still a great body of work—I might instance the calculation of cash bonuses when these are a certain percentage of reserve-values, premiums paid, loading, or the like, and the conversion of such cash bonuses into reversions when policies are grouped according to the present age of the life assured—for which Crelle's Tables afford unequalled facilities, and they must always remain an invaluable addition to the computer's library.

It adds to the interest, though not necessarily to the value, of the new method, to observe that its results are obtained by means of a single column of figures, occupying very much the same space as each one of Crelle's 900 columns, and costing only a small fraction of the price of Crelle's book.

17. So far I have confined myself to exhibiting the working of the new method with reference to calculations which occur in life assurance offices, and I have done so for various reasons: partly because actuarial work furnishes, probably, the best field for all calculating processes, and is, in the main, the best test of their value; partly because it is in this connection that I have myself employed the method; and partly because I fear it will be impossible to get the general public to adopt it, however clearly it may be shown to be to their advantage so to do. There is a prevailing belief that logarithms are a peculiarly abstruse branch of mathematical science, and when people learn that this method is based on logarithms, they will be apt at once to decline its further acquaintance, modestly pleading their inability to understand its working!

It is obvious, however, that there are a great many operations of a general character to which the method might be very usefully applied. Questions dealing with the conversion into English of French measures of length, surface, capacity or weight; questions of exchange: questions involving a combination of foreign measures and foreign moneys—for all such calculations the new method will be found very convenient, when, as is often the case, a near approximation is all that is required. Each individual user of the method will have one or two operations of this kind occurring in his own practice, and it would be a simple matter for him to note on the card the appropriate logarithms. Thus, if he were accustomed to get prices quoted in francs per 1,000 kilogrammes, and he wanted to arrive rapidly at a rough idea of these prices stated in £ per ton, his multiplier would be (taking the sterling value of the franc as $8\frac{3}{4}d.$) $\frac{\cdot 03646}{\cdot 9842}$, and the logarithm of

this fraction—569—would denote the number of spaces in advance of the given price at which he would find the required result, his knowledge of his own trade being a sufficient guide to him in the matter of the decimal point.

For one more example under this head we may refer to Table I (Art. 15). If a small subsidiary table be made out giving the logarithms to three places of, say, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, and 5, this will enable us to use the logarithms of days of the year as a simple interest table at these rates. All we have to do is to advance from the principal the number of spaces indicated by the logarithm of the rate, *in addition to* the number set down opposite the given number of days. Deduction of income tax may also be managed in a very simple way. If the tax is at the rate of t pence per £, then the net interest will be equal to the gross interest multiplied by $\frac{240-t}{240}$. It is a very easy matter to record the logarithms of this function for the usual rates of tax, and then these logarithms will denote the final advance which has to be made in order to reach the net interest. These subsidiary tables might be noted on the card in this form:

Table III.

Interest		Income Tax	
Rate per-cent	Spaces in Advance	Rate per £	Spaces in Advance
3	477	4*d.*	993
$3\frac{1}{2}$	544	5*d.*	991
4	602	6*d.*	989
$4\frac{1}{2}$	653	7*d.*	987
5	699	8*d.*	985

Suppose, for instance, that it be required to calculate the interest on £125. 6*s.* 3*d.* for 130 days at 3 per-cent and to deduct income tax at 6*d.* per £. We are now in a position to do this by a purely mechanical operation, and without setting down any intermediate figures. Placing one finger of the left hand on 1253 (the nearest number to the principal) we advance 552 spaces (the number opposite to 130 days in Table I), which brings us up to 4467; from this point we advance 477 spaces (the number opposite 3 per-cent in Table III), which brings us up to 1340; and from this point we go backwards 11 spaces (the arithmetical

complement of the number opposite 6*d*. per £ in Table III), which brings us up to 1306. We then set down the result as £1. 6*s*. 1*d*., which will be found to be correct. If we had added mentally the first two numbers of spaces in advance (552 + 477 = 1029), we should have saved ourselves a little trouble, as we should have had to advance only 29 spaces from our original starting point to reach the gross interest. Such a process, however, would involve some liability to error, and in adopting it we disturb to some extent the purely mechanical character of the operation; but if several similar calculations had to be performed it might be usefully adopted in respect of the last two steps, as the rates of interest and tax would be uniform throughout and the sum of the corresponding logarithms might be written down before beginning the work. This method of calculating interest would be only applicable, of course, to cases involving small amounts, and the illustration has been given mainly as a suggestion of how the card may be employed in combining a number of operations.

18. In conclusion it may be pointed out that the principle underlying the new method might be extended so as to apply to larger collections of anti-logarithms. If, for instance, it were desired to use Hannyngton's Four Place Table in this way, it would be possible to give effect to *four* figures of the logarithm of the multiplier, *pages* being turned over for the first figure, *groups* of 100 spaces advanced for the second figure, *lines* of 10 spaces for the third figure, and *spaces* for the fourth figure. Similarly with regard to Erskine Scott's Table of Anti-logarithms. Here *five* figures could be made use of, *pages* being turned over for the first two figures, the third, fourth and fifth figures being dealt with in the same way as the second, third and fourth in the case of Hannyngton's Table. It is not claimed, however, that any particular value attaches to this method of procedure. When pages have to be turned over, the process becomes much more tedious: it requires time to find the position of the multiplicand, and it is difficult to place the pen on the corresponding point a certain number of pages in advance.*

* Erskine Scott's book may sometimes be employed with tolerable ease in drawing up Sinking Fund Tables, and will give good results when each instalment is under £100. Log $\left(1 + \frac{i}{m}\right)$ can hardly ever amount to ·02, so that the first step will not require more than one page to be turned over, and as the process is a continuous one the difficulty of finding the position of the multiplicands will not arise. If each tenth value be first calculated by way of check, the position of the anti-logarithm may be more accurately adjusted at these points.

It would appear that, in order to obtain whatever advantages the new method may afford, its application must be mainly restricted to that arrangement of the anti-logarithm table which brings all our material before us at one view.

Discussion on Mr. Hardy's and Mr. Robertson's Papers.

The President (Mr. A. J. Finlaison, C.B.) said Mr. Hardy had not discussed the general subject of extra risks, nor the method of providing for such risks by temporary deductions from the amount assured, which had been ably attacked in 1891 by the late Mr. Sunderland, but had assumed that the amount of extra premium charged was exactly commensurate with the extra risk sustained from time to time—a view of the problem which would doubtless commend itself to Mr. Bailey, who had frequently advocated the treatment of extra premiums in a manner that would correspond to the suggested assumption. The convenience of the assumptions was, as the author showed, obvious. The paper which they welcomed from Mr. Robertson dealt also with an instrument for the performance of calculations. Mr. Robertson had shown how a table of logarithms might be viewed as an extended slide rule, and that a card of four-figure logarithms or anti-logarithms could be practically used as a slide rule. Whether the suggested method was superior to the ordinary way of using the table, could only be determined by trial. It might prove attractive to some computers and afford relief to others by varying the mental strain produced by protracted calculations of an identical kind. He would invite Mr. Chisholm to open the discussion.

Mr. Jas. Chisholm said the last paper submitted to the Institute on the subject of extra premiums was that read in 1891 by the late Mr. Sunderland, and the first speaker on Mr. Sunderland's paper was the late Mr. Rothery. It seemed impossible to approach now the criticism of the same subject, without paying a tribute again to the memory of these two gifted members of the Institute removed from amongst them almost at the threshold of their career. Life assurance was based on the principle of broad averages, and there were often necessarily somewhat considerable differences in the risks admitted at the ordinary rate—lives drawn from different parts of the country, healthy and unhealthy, following occupations of widely differing character, and reported on by medical examiners more or less experienced. The range of selection was so wide that it became a most hard and invidious task to draw the line and say, on this side there should be admission at the ordinary rate, and on the other a considerable extra premium would be imposed: for it was the general opinion that a small and irritating amount of extra was worse than useless. Then there were questions that might be raised as to the incidence of the extra risk, the different amount of risk under the various classes of policies, the surrender-values to be allowed and the amount of reserves to be held. In his interesting and practical paper Mr. Hardy had excluded all these questions by the conditions

Table of Anti-Logarithms, *referred to on page* 166.

	0	1	2	3	4	5	6	7	8	9	
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	68
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	69
70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	70
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	71
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	72
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	73
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	74
75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	75
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	76
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	77
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	78
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	79
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	80
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	81
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	82
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	83
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	84
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	85
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	86
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	87
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	88
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	89
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	90
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	91
92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	92
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	93
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	94
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	95
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	96
97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	97
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	98
99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	99

		0	1	2	3	4	5	6	7	8	9		
0	0	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0
1	1	[illegible]	1026	1028	1030	1033	1035	1038	1040	1042	1047	1	1
2	2	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	2	2
3	3	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	3	3
4	4	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	4	4
5	5	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	5	5
6	6	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	6	6
7	7	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	7	7
8	8	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	8	8
9	9	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	9	9
10	10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	10	10
11	11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	11	11
12	12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	12	12
13	13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	13	13
14	14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	14	14
15	15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	15	15
16	16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	16	16
17	17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	17	17
18	18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	18	18
19	19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	19	19
20	20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	20	20
21	21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	21	21
22	22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	22	22
23	23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	23	23
24	24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	24	24
25	25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	25	25
26	26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	26	26
27	27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	27	27
28	28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	28	28
29	29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	29	29
30	30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	30	30
31	31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	31	31
32	32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	32	32
33	33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	33	33
34	34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	34	34
35	35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	35	35
36	36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	36	36
37	37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	37	37
38	38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	38	38
39	39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	39	39
40	40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	40	40
41	41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	41	41
42	42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	42	42
43	43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	43	43
44	44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	44	44
45	45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	45	45
46	46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	46	46
47	47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	47	47
48	48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	48	48
49	49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	49	49
50	50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	50	50
51	51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	51	51
52	52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	52	52
53	53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	53	53
54	54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	54	54
55	55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	55	55
56	56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	56	56
57	57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	57	57
58	58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	58	58
59	59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	59	59
60	60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	60	60
61	61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	61	61
62	62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	62	62
63	63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	63	63
64	64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	64	64
65	65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	65	65
66	66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	66	66
67	67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	67	67
68	68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	68	68
69	69	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	69	69
70	70	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5117	70	70
71	71	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	71	71
72	72	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	72
73	73	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	73	73
74	74	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	74	74
75	75	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	75	75
76	76	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	76	76
77	77	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	77	77
78	78	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	78	78
79	79	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	79	79
80	80	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6412	[illegible]	[illegible]	80	80
81	81	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	81	81
82	82	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	82	82
83	83	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	83	83
84	84	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	84	84
85	85	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	85	85
86	86	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	86	86
87	87	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	87	87
88	88	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	88	88
89	89	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	89	89
90	90	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	90	90
91	91	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91	91
92	92	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	92	92
93	93	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	93	93
94	94	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	94	94
95	95	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9016	[illegible]	[illegible]	[illegible]	[illegible]	95	95
96	96	[illegible]	[illegible]	[illegible]	[illegible]	9204	9226	9247	[illegible]	[illegible]	[illegible]	96	96
97	97	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	97	97
98	98	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	98	98
99	99	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9977	99	99

with which he set out, and by the novel manner in which he approached the problem. Instead of starting with various assumed amounts of extra premium and trying to find out what were the proper deductions from the sum assured as the equivalent of these extras, he assumed certain fixed amounts of initial deduction, and then framed a table of extras of which these were the equivalent. As a result he was able to stipulate from the beginning that the deductions were intended to be applied only in those cases where the extra mortality did not extend beyond what was known as the expectation of life; that was to say, those cases chiefly in which the extra risk arose from a consumptive family history. It was just this class of case to which the deduction method was suitable, and it was also the class in which there was the greatest difficulty in persuading a person who was personally healthy that it was reasonable to pay an extra at all. Here, then, they had a practical difficulty met in a way which enabled the assured to take his own risk over the period that it was assumed to last. The author's method would also seem to imply that the surrender-value to be allowed was the same as that of an ordinary policy, and he (Mr. Chisholm) thought that in practice it was impossible to give a man who in any form paid an extra, a smaller surrender-value than if the ordinary premium only had been charged. The paper was an admirable example of simple and concise work. He should have liked Mr. Hardy to have given more information in detail as to the interpolations in Table II, for the functions to be dealt with were somewhat irregular on account of the varying term over which the deductions were to be spread. There was a general agreement that, speaking of the extras as so many years addition to the age, was a false nomenclature. The incidence of the mortality risk of a rated-up life could not be assumed to be the same as that of a select life at the rated-up age. He feared however there was no other course open to them as they were not likely ever to have a complete mortality table for every description of extra risk. Still, he preferred to speak of an "extra" as so many shillings and pence added to the premium. In doing this they would not run the risk of deluding themselves into the belief that extras were ever arrived at by other than an arbitrary process, and would avoid the affectation of precision involved in calling a man five or seven years older than he really was. He (Mr. Chisholm) had recast Table IV of the paper in the form of deductions equivalent to fixed extras of 5*s*., 10*s*., 15*s*., and £1, and for the sake of comparison had added the similar deductions published in 1889 by his colleague, Dr Pollock, and himself in a table in the Medical Handbook of Life Assurance. These deductions were arrived at by a totally different process from that of Mr. Hardy's. They were the equivalent in value at the rated-up age, by the H^M mortality and 4 per-cent interest, of the fixed amounts of extra imposed, valued also at the rated-up age. The two sets of deductions were practically identical at nearly all ages, where Mr. Hardy's results were given. The similarity was probably explainable by the fact that as they were dealing only with differences in rates of premium for various ages, a change of $\frac{1}{2}$, or even 1 per-cent, in the rate of interest did not make much alteration. He would have liked to

add to the table Mr. Sunderland's results also, had the values calculated enabled him to do so. They did not, however, differ much from Mr. Hardy's, although in the one case a non-profit assurance was dealt with, and in the other a with-profit assurance with a £2 per-cent bonus.

Age at Entry	Expectation of Life	Deductions from Sum Assured Corresponding to Fixed Extras							
		5s.		10s.		15s.		20s.	
		Hardy	Pollock and Chisholm	Hardy	Pollock and Chisholm	Hardy	Pollock and Chisholm	Hardy	Pollock and Chisholm
20	42	47	38	62	61	—	76	—	86
25	38	41	36	57	61	—	76	—	86
30	35	35	33	53	55	—	71	—	81
35	31	31	30	49	51	—	66	—	75
40	27	27	27	45	46	56	60	—	72
45	24	23	22	40	39	51	51	—	61
50	20	20	19	35	33	45	45	55	55
55	17	—	15	29	27	38	38	47	47
60	14	—	12	23	21	31	30	39	38

Mr. Stanley Day referred to the close agreement of Mr. Sunderland's results and those of the author. When the rating-up was five years they were practically identical,—but in the rating-up ten years he found the deductions made by Mr. Sunderland were about 10 per-cent in excess of those made by Mr. Hardy. In his own practice a rough and ready method had been adopted, but it seemed to work out very fairly. For the ordinary fixed debt, he took in the first place the difference between the assurance that would be covered by the proper premium at the rated-up age and the assurance that was covered by the true-age premium at the rated-up age, and increased the amount by a third on account of the debt ceasing on the "expectation of life" being reached. For the initial amount of the diminishing debt, he doubled the corresponding fixed debt. Perhaps one or two examples would be interesting. For 5 years' extra the debt was £31 per-cent at age at entry 20. Mr. Sunderland made it £31, and Mr. Hardy £34. At age 50 at entry, and 5 years' surcharge, the diminishing debt with him was £49, Mr. Sunderland's was £51, and Mr. Hardy's £50. With the rating-up at 10 years, his figures agreed closely with Mr. Sunderland's, but not with the author's. Perhaps Mr. Hardy could explain how the difference arose.

Mr. W. Sutton, speaking of Mr. Robertson's paper, said he (Mr. Sutton) was personally in a most unhappy position. He had never been a believer in mechanical appliances for actuarial calculations. The paper must have given its author a great deal of trouble and

labour, although this was not so apparent on the surface. One of the difficulties in the new method he found was in counting the spaces, although the author seemed to have done his best to make that as light as possible. The explanation of the process was very clear, but he was afraid it would rest with the next generation to decide as to its practical use.

Mr. A. F. BURRIDGE said one or two rules might be laid down within the limits of which mechanical aids were permissible. First, if by their assistance work was lightened and accurate results were given. Secondly, they must be easy of application, for, as the President had said, a computer who had long been accustomed to one particular method did not easily acquire a different one. Thirdly, and he could speak from experience, if they lessened the fatigue to brain and eye in lengthy operations. So far as he had looked into the paper the method fulfilled, within the modest limits which the author had assigned to it, nearly all those conditions. He (Mr. Burridge) had applied it in a series of calculations at first hand, and not simply to the checking process. There were other methods in use in the office, but this method was chosen purposely, and he would tell Mr. Robertson that it had proved extremely successful. As a referee he had felt bound to surround himself with various uncanny instruments, and he had before him all the various instruments referred to in the paper. Amongst them was the original slide rule on which the whole system of the paper and Gunter's line was based, also Professor Fuller's Slide Rule and the Patent Calculator of M. Boucher, of Paris. Gunter's was a long line divided into 1,000 unequal parts proportional to the logarithms of the natural numbers. The author's, on the other hand, was a long line divided into 1,000 equal parts, the natural numbers corresponding to the equi-distant logarithms being placed against the divisions. The system might be a useful check when only three or four figures in the result were necessary, but nothing more than four figures could be obtained. The conclusion at which he had therefore arrived was that it was a most interesting departure to have some such table of anti-logarithms to work in connection with other systems of computation. The author limited the use of this method to checking purposes, and he was right probably in so doing. As this was essentially a paper on slide rules he would draw attention to one in which he was more interested from long use than any other. He had had the pleasure of introducing the Fuller slide rule to the Institute as long ago as 1882. Mr. Robertson had dismissed Fuller in rather a few words, and perhaps on that account, if on no other, he might be allowed to draw attention to the great use of this particular form of rule. The number of actual figures recorded, including spaces representing figures, was 7,250, *i.e.*, every number from 1,010 to 6,500, and after that every alternate number. In Mr. Robertson's system only 1,000 actual numbers were given. Fuller's scale, therefore, was exactly like Gunter's line, except that instead of being a straight line requiring a long pair of compasses, it was wound round a cylinder. He believed they would see the new system in work, and hoped that this discussion would be the means of interesting members in the use of these methods, not omitting his favourite Fuller.

The PRESIDENT having proposed a vote of thanks to the authors, which was unanimously accorded,

Mr. G. F. HARDY, in reply, said he had not attempted to deal with the subject generally, because that had been thoroughly discussed a short time since. He was much interested in the examples given by Mr. Day, and the extremely near results obtained by his (Mr. Day's) simple method to those obtained by Mr. Sunderland's and his own somewhat more complicated methods. The difference between Mr. Sunderland's figures—when 10 years' addition was made to the age—and his own, might be due to the fact, that, throwing a greater part of the extra mortality into the earlier years of assurance rather than into the later years, naturally gave a greater weight to deductions from the sum assured, and Mr. Sunderland's method dealt with the extra risk as though the mortality was spread over the whole of life in proportion to the difference between the rated-up age and the true age. With regard to Mr. Chisholm's reference to the term "rating-up," he was certainly theoretically right. The idea of adding so many years to the age was no doubt responsible for the further notion that the life once rated up 10 years was always to be represented by a life 10 years older than the true age, which was of course an unsatisfactory supposition. It had probably arisen from the necessity of having some simple method of classifying lives which the medical examiners could understand and use. The method with which he first had any acquaintance was, that the medical examiner stated how many years he considered should be deducted from the "expectation of life"; and that was a very general method still. What affected the insurer was that there should be an addition of so much to the premium, but in writing a paper which was almost entirely theoretical the question, of course, would arise, what scale of premium should be employed? If they dealt with rating-up, *i.e.*, additions of so many years to the age, then of course they meant an addition of certain specific sums to the net premium. Taking the H^M Table for example and rating a life up from 20 to 30 they knew that that meant exactly an addition of so much to the net premium, but in practice they had to deal with office premiums, which were loaded in various ways, and therefore the addition of 5*s*. or 10*s*. per-cent in one office did not quite represent the sum surcharged in another. It would seem better therefore to deal with net premiums than to rate-up the ages.

Mr. ROBERTSON said he was much interested in what Mr. Burridge had stated as to the employment of his method. It would sometimes happen that from the character of the particular multiplier the table would be found especially convenient; for instance, when, as Mr. Burridge had said, the logarithm of the multiplier was ·967. Then of course the operation to be performed was extremely slight, the required result being obtained at once by referring back 33 spaces. Mr. Burridge spoke of four figures in the results being given by the new method, and for the purpose of a check it would often, no doubt, be sufficiently accurate to that extent.

ACTUARIAL NOTE.

On the Calculation of Contingent Assurances. *By* FRANCIS E. COLENSO, M.A., F.I.A., *Actuary of the Eagle Insurance Company.*

IN a paper which was read by me before the Institute on the 30 April 1894 (*J.I.A.*, xxxi, 337), it is shown that when a mortality table has been graduated by Makeham's formula $\mu_x = -\lambda s - \lambda g \lambda c c^x$, the numerical values of contingent assurances may be calculated by means of the relation

$$\bar{A}^1_{xyz\ldots} = -\lambda s \bar{a}_{xyz\ldots} + (\mu_x + \lambda s)\bar{a}'_{xyz\ldots} \quad . \quad . \quad . \quad (1)$$

the symbol $\bar{a}'_{xyz\ldots}$ denoting an annuity calculated at a rate of interest i' such that $\frac{1}{1+i'} = vc$.

Analogous relations were shown to subsist in the case of certain other survivorship benefits, and tables were submitted to illustrate the application of such formulas.

After considering a little further the expression (1) for $\bar{A}^1_{xyz\ldots}$, I have arrived at a formula which reduces the computation of this benefit to a still more simple operation.

Since

$$(\delta - n\lambda s)a_{xyz\ldots} = 1 - n(\mu_r + \lambda s)\bar{a}'_{xyz\ldots},$$

we may, by means of this relation, eliminate $a_{xyz\ldots}$ from (1), and write

$$\bar{A}^1_{xyz\ldots} = \kappa_n - \{\mathbf{f}_{rrr\ldots} - (\mu_x + \lambda s)\}\bar{a}'_{rrr\ldots} \quad . \quad . \quad (2)$$

where κ_n, for a given number n of lives, is a constant $= \frac{\lambda s}{n\lambda s - \delta}$, r is the equalized age, and $\mathbf{f}_{rrr\ldots} = \kappa_n \times n(\mu_r + \lambda s)$, so that this last function may be tabulated and used in the place of μ to determine the equalized age r.

The numerical labour involved in the use of this formula may be illustrated by the following example:—

Required $\bar{A}^1_{50:30:40:60}$.

Here $n = 4$, and $\kappa_n = \cdot 13185$.

We have only to find $\bar{a}'_{50:30:40:60}$ by the usual process, employing, as stated, $\Sigma\mathbf{f}$, instead of $\Sigma\mu$, to find r, the equalized age.

$$\text{Thus}\quad \mathbf{f}_{30:30:30:30}+\mathbf{f}_{40:40:40:40}+\&\text{c.}= \cdot004109=4\mathbf{f}_{rrrr}.$$

$$\therefore\ r=50\cdot144.\qquad \text{Also}\ \mu_{50}+\lambda s= \cdot007687$$

$$\text{Deducting, we get}\quad -\cdot003578$$

$$\text{and multiplying this by}\ \bar{a}'_{rrrr} = 16\cdot177$$

$$\text{The product}=-\cdot05788$$

$$\kappa_4= \cdot13185$$

$$\therefore\ \bar{A}^{1}_{50:30:40:60}= \cdot18973$$

If the life "at risk" be any other of the lives, as 60, the only variation in the above figures is that instead of $\mu_{50}+\lambda s$ we deduct $\mu_{60}+\lambda s$.

It follows that in the case of $\bar{A}_{\overset{1}{\overline{50:60:30}}:40}$ say, we deduct $\frac{1}{3}(\mu_{50}+\mu_{60}+\mu_{30}+3\lambda s)$, and taking, as before, our product from κ_4, get $\frac{1}{3}\bar{A}_{\overset{1}{\overline{50:60:30}}:40}$.

In the preparation of the accompanying tables (*see J.I.A.*, xxxi, 343) the various columns have been carefully checked by casting and differencing, and every effort has been made to render the set an accurate, as it is a compact, instrument for computing, in cases where not more than four lives are involved, the values of joint-life and survivorship benefits at 3 per-cent interest upon the basis of the Carlisle Mortality.

I take the opportunity of calling attention to a few errors which have escaped correction in the above-cited paper.

p. 343, line 21. Before $\frac{1}{12}\Delta\mu$ insert $\Delta a+$.

p. 344, line 11. For $h'(\bar{a}'_{y|xz}+\bar{a}'_{x|yz})$
read $h_x\bar{a}'_{y|xz}+h_y\bar{a}'_{x|yz}$, where $h_x=\mu_x+\lambda s$.

p. 347, line 13. The factor outside the integral should be l_x^{-1}.

,, line 19. For xy read yz.

p. 348, line 10 from foot. For $\bar{e}_{yz}.\bar{a}_{xyz}$ read $\bar{e}_{yz}.\bar{a}_x$.

p. 349, line 3 from foot. The reference should be to—
(4) *supra*. p. 341.

Carlisle Table—3 *per-cent.*

x	$\mu_x + \lambda s$	$\log_{10} l_x$	$\bar{a}_x$	x	$\mu_x + \lambda s$	$\log_{10} l_x$	$\bar{a}_x$
15	·000314	4·80596	22·745	58	·015962	4·57755	11·978
16	·000344	·80223	22·610	59	·017489	·56671	11·621
17	·000377	·79849	22·470	60	·019161	·55518	11·263
18	·000413	·79474	22·326	61	·020994	·54288	10·904
19	·000453	·79097	22·177	62	·023002	·52975	10·546
20	·000496	·78718	22·023	63	·025202	·51571	10·187
21	·000544	·78338	21·864	64	·027612	·50067	9·829
22	·000596	·77955	21·700	65	·030253	·48453	9·473
23	·000653	·77570	21·530	66	·033146	·46719	9·119
24	·000715	·77182	21·356	67	·036316	·44854	8·768
25	·000784	·76791	21·175	68	·039789	·42844	8·419
26	·000858	·76397	20·990	69	·043595	·40677	8·074
27	·000941	·76000	20·798	70	·047764	·38336	7·734
28	·001031	·75599	20·601	71	·052332	·35806	7·398
29	·001129	·75194	20·399	72	·057337	·33068	7·068
30	·001237	·74785	20·190	73	·062821	·30102	6·743
31	·001355	·74370	19·975	74	·068829	·26887	6·425
32	·001485	·73951	19·754	75	·075411	·23399	6·113
33	·001627	·73525	19·527	76	·082624	·19612	5·809
34	·001783	·73093	19·294	77	·090526	·15496	5·513
35	·001953	·72654	19·055	78	·099183	·11021	5·224
36	·002140	·72207	18·809	79	·108669	·06153	4·944
37	·002345	·71751	18·558	80	·119062	·00853	4·673
38	·002569	·71286	18·300	81	·130449	3·95080	4·411
39	·002815	·70811	18·035	82	·142925	·88790	4·157
40	·003084	·70325	17·765	83	·156594	·81932	3·913
41	·003379	·69827	17·488	84	·171570	·74453	3·679
42	·003702	·69315	17·205	85	·187979	·66292	3·454
43	·004056	·68789	16·917	86	·205957	·57386	3·238
44	·004444	·68246	16·622	87	·225654	·47662	3·033
45	·004869	·67686	16·321	88	·247235	·37042	2·836
46	·005334	·67106	16·015	89	·270880	·25449	2·650
47	·005844	·66506	15·703	90	·296787	·12764	2·472
48	·006403	·65882	15·386	91	·325171	2·98909	2·304
49	·007016	·65232	15·063	92	·356269	·83763	2·145
50	·007687	·64555	14·736	93	·390342	·67204	1·994
51	·008422	·63847	14·404	94	·427674	·49094	1·853
52	·009228	·63106	14·068	95	·468576	·29287	1·719
53	·010110	·62328	13·728	96	·513390	·07620	1·594
54	·011077	·61511	13·384	97	·562489	1·83916	1·476
55	·012136	·60649	13·036	98	·616285	·57978	1·366
56	·013297	·59739	12·686	99	·675225	·29594	1·264
57	·014569	·58776	12·333	100	·739802	0·98529	1·169

$\text{Log}_e s = -·0082462$ $\text{Log}_e c = ·0913369$ $\text{Log}_e g = -·0008745$

Carlisle Table—3 *per-cent.*

x	f_{xx}	Δ	$(\Delta)^{-1}$	x	f_{xx}	Δ	$(\Delta)^{-1}$
15	·000056	·000006	167	58	·002858	·000274	365
16	·000062	·000006	167	59	·003132	·000299	334
17	·000068	·000006	167	60	·003431	·000328	305
18	·000074	·000007	143	61	·003759	·000360	278
19	·000081	·000008	125	62	·004119	·000394	254
20	·000089	·000008	125	63	·004513	·000431	232
21	·000097	·000010	100	64	·004944	·000473	211
22	·000107	·000010	100	65	·005417	·000518	193
23	·000117	·000011	910	66	·005935	·000568	176
24	·000128	·000012	833	67	·006503	·000622	161
25	·000140	·000014	714	68	·007125	·000681	147
26	·000154	·000014	714	69	·007806	·000747	134
27	·000168	·000016	625	70	·008553	·000818	122
28	·000184	·000018	556	71	·009371	·000896	112
29	·000202	·000020	500	72	·010267	·000982	102
30	·000222	·000021	476	73	·011249	·001076	929
31	·000243	·000023	435	74	·012325	·001179	848
32	·000266	·000025	400	75	·013504	·001291	775
33	·000291	·000028	357	76	·014795	·001415	707
34	·000319	·000031	323	77	·016210	·001550	645
35	·000350	·000033	303	78	·017760	·001699	589
36	·000383	·000037	270	79	·019459	·001861	537
37	·000420	·000040	250	80	·021320	·002039	490
38	·000460	·000044	227	81	·023359	·002234	448
39	·000504	·000048	208	82	·025593	·002448	408
40	·000552	·000053	189	83	·028041	·002681	373
41	·000605	·000058	172	84	·030722	·002939	340
42	·000663	·000063	159	85	·033661	·003219	311
43	·000726	·000070	143	86	·036880	·003527	284
44	·000796	·000076	132	87	·040407	·003864	259
45	·000872	·000083	120	88	·044271	·004234	236
46	·000955	·000092	109	89	·048505	·004639	215
47	·001047	·000100	100	90	·053144	·005083	197
48	·001147	·000109	917	91	·058227	·005569	180
49	·001256	·000120	833	92	·063796	·006101	164
50	·001376	·000132	758	93	·069897	·006685	150
51	·001508	·000144	694	94	·076582	·007324	137
52	·001652	·000158	633	95	·083906	·008025	125
53	·001810	·000174	575	96	·091931	·008792	114
54	·001984	·000189	529	97	·100723	·009633	104
55	·002173	·000208	481	98	·110356	·010554	948
56	·002381	·000228	439	99	·120910	·011563	865
57	·002609	·000249	402	100	·132473	...	...

$k_2 = \cdot 17907$

CARLISLE TABLE—3 *per-cent.*

x	$\bar{a}'_{xx}$	$-\Delta\bar{a}'_{xx}$	$\bar{a}_{xx}$	$-\Delta\bar{a}_{xx}$	x	$\bar{a}'_{xx}$	$-\Delta\bar{a}'_{xx}$	$\bar{a}_{xx}$	$-\Delta\bar{a}_{xx}$
15	210·424	10·187	18·843	·123	58	18·696	1·216	8·755	·317
16	200·237	9·687	18·720	·127	59	17·480	1·146	8·438	·315
17	190·550	9·279	18·593	·133	60	16·334	1·082	8·123	·314
18	181·271	8·841	18·460	·137	61	15·252	1·020	7·809	·311
19	172·430	8·452	18·323	·142	62	14·232	·961	7·498	·308
20	163·978	8·089	18·181	·147	63	13·271	·904	7·190	·305
21	155·889	7·721	18·034	·153	64	12·367	·851	6·885	·301
22	148·168	7·358	17·881	·158	65	11·516	·800	6·584	·296
23	140·810	7·025	17·723	·163	66	10·716	·751	6·288	·291
24	133·785	6·712	17·560	·169	67	9·965	·704	5·997	·285
25	127·073	6·401	17·391	·175	68	9·261	·661	5·712	·279
26	120·672	6·115	17·216	·180	69	8·600	·619	5·433	·273
27	114·557	5·840	17·036	·187	70	7·981	·580	5·160	·266
28	108·717	5·572	16·849	·192	71	7·401	·542	4·894	·259
29	103·145	5·314	16·657	·198	72	6·859	·507	4·635	·251
30	97·831	5·067	16·459	·205	73	6·352	·473	4·384	·243
31	92·764	4·832	16·254	·210	74	5·879	·441	4·141	·235
32	87·932	4·610	16·044	·217	75	5·438	·412	3·906	·227
33	83·322	4·397	15·827	·222	76	5·026	·383	3·679	·218
34	78·925	4·193	15·605	·229	77	4·643	·357	3·461	·210
35	74·732	3·996	15·376	·235	78	4·286	·331	3·251	·201
36	70·736	3·809	15·141	·241	79	3·955	·308	3·050	·193
37	66·927	3·628	14·900	·247	80	3·647	·287	2·857	·183
38	63·299	3·456	14·653	·253	81	3·360	·264	2·674	·175
39	59·843	3·293	14·400	·259	82	3·096	·246	2·499	·167
40	56·550	3·134	14·141	·264	83	2·850	·228	2·332	·158
41	53·416	2·984	13·877	·270	84	2·622	·210	2·174	·149
42	50·432	2·839	13·607	·275	85	2·412	·195	2·025	·141
43	47·593	2·701	13·332	·280	86	2·217	·180	1·884	·134
44	44·892	2·569	13·052	·286	87	2·037	·166	1·750	·126
45	42·323	2·442	12·766	·290	88	1·871	·153	1·624	·118
46	39·881	2·321	12·476	·294	89	1·718	·142	1·506	·111
47	37·560	2·205	12·182	·299	90	1·576	·130	1·395	·104
48	35·355	2·094	11·883	·303	91	1·446	·120	1·291	·097
49	33·261	1·987	11·580	·305	92	1·326	·110	1·194	·091
50	31·274	1·886	11·275	·309	93	1·216	·102	1·103	·085
51	29·388	1·789	10·966	·312	94	1·114	·094	1·018	·079
52	27·599	1·693	10·654	·314	95	1·020	·085	·939	·074
53	25·906	1·606	10·340	·315	96	·935	·079	·865	·069
54	24·300	1·520	10·025	·317	97	·856	·072	·796	·064
55	22·780	1·438	9·708	·318	98	·784	·066	·732	·059
56	21·342	1·361	9·390	·317	99	·718	·061	·673	·055
57	19·981	1·285	9·073	·318	100	657	...	·618	...

Carlisle Table—3 *per-cent.*

x	f_{xxx}	Δ	$(\Delta)^{-1}$	x	f_{xxx}	Δ	$(\Delta)^{-1}$
15	·000048	·000004	250	58	·002424	·000232	431
16	·000052	·000005	200	59	·002656	·000254	394
17	·000057	·000006	167	60	·002910	·000278	360
18	·000063	·000006	167	61	·003188	·000305	328
19	·000069	·000006	167	62	·003493	·000334	299
20	·000075	·000008	125	63	·003827	·000366	273
21	·000083	·000008	125	64	·004193	·000402	249
22	·000091	·000008	125	65	·004595	·000439	228
23	·000099	·000010	100	66	·005034	·000481	208
24	·000109	·000010	100	67	·005515	·000528	189
25	·000119	·000011	910	68	·006043	·000578	173
26	·000130	·000013	769	69	·006621	·000633	158
27	·000143	·000014	714	70	·007254	·000694	144
28	·000157	·000015	667	71	·007948	·000760	132
29	·000172	·000016	625	72	·008708	·000833	120
30	·000188	·000018	556	73	·009541	·000912	110
31	·000206	·000020	500	74	·010453	·001000	100
32	·000226	·000021	476	75	·011453	·001095	913
33	·000247	·000024	417	76	·012548	·001200	833
34	·000271	·000026	385	77	·013748	·001315	760
35	·000297	·000028	357	78	·015063	·001441	694
36	·000325	·000031	323	79	·016504	·001578	634
37	·000356	·000034	303	80	·018082	·001729	578
38	·000390	·000037	270	81	·019811	·001895	528
39	·000427	·000041	244	82	·021706	·002076	482
40	·000468	·000045	222	83	·023782	·002275	440
41	·000513	·000049	204	84	·026057	·002492	401
42	·000562	·000054	185	85	·028549	·002730	366
43	·000616	·000059	169	86	·031279	·002991	334
44	·000675	·000064	156	87	·034270	·003278	305
45	·000739	·000071	141	88	·037548	·003591	278
46	·000810	·000078	128	89	·041139	·003934	254
47	·000888	·000085	118	90	·045073	·004311	232
48	·000973	·000093	108	91	·049384	·004723	212
49	·001066	·000101	990	92	·054107	·005175	193
50	·001167	·000112	893	93	·059282	·005669	176
51	·001279	·000122	820	94	·064951	·006212	161
52	·001401	·000134	746	95	·071163	·006806	147
53	·001535	·000147	680	96	·077969	·007457	134
54	·001682	·000161	621	97	·085426	·008170	122
55	·001843	·000176	568	98	·093596	·008951	112
56	·002019	·000194	515	99	·102547	·0[illegible]9807	102
57	·002213	·000211	474	100	·112354	..	...

$k_3 = \cdot 15187$

CARLISLE TABLE—3 *per-cent.*

Age	$\bar{a}'_{xxx}$	$-\Delta\bar{a}'_{xxx}$	$\bar{a}_{xxx}$	$-\Delta\bar{a}_{xxx}$	Age	$\bar{a}'_{xxx}$	$-\Delta\bar{a}'_{xxx}$	$\bar{a}_{xxx}$	$-\Delta\bar{a}_{xxx}$
15	127·409	5·509	16·205	·107	58	12·915	·837	7·027	·281
16	121·900	5·276	16·098	·112	59	12·078	·791	6·746	·278
17	116·624	5·094	15·986	·116	60	11·287	·748	6·468	·276
18	111·530	4·890	15·870	·121	61	10·539	·706	6·192	·272
19	106·640	4·712	15·749	·126	62	9·833	·666	5·920	·267
20	101·928	4·544	15·623	·131	63	9·167	·627	5·653	·264
21	97·384	4·364	15·492	·136	64	8·540	·591	5·389	·258
22	93·020	4·201	15·356	·142	65	7·949	·556	5·131	·253
23	88·819	4·035	15·214	·147	66	7·393	·522	4·878	·247
24	84·784	3·883	15·067	·152	67	6·871	·490	4·631	·241
25	80·901	3·729	14·915	·158	68	6·381	·460	4·390	·234
26	77·172	3·589	14·757	·164	69	5·921	·431	4·156	·228
27	73·583	3·453	14·593	·169	70	5·490	·403	3·928	·220
28	70·130	3·316	14·424	·175	71	5·087	·377	3·708	·213
29	66·814	3·186	14·249	·181	72	4·710	·352	3·495	·205
30	63·628	3·058	14·068	·187	73	4·358	·328	3·290	·197
31	60·570	2·936	13·881	·193	74	4·030	·307	3·093	·189
32	57·634	2·819	13·688	·198	75	3·723	·285	2·904	·182
33	54·815	2·707	13·490	·205	76	3·438	·265	2·722	·173
34	52·108	2·598	13·285	·211	77	3·173	·247	2·549	·166
35	49·510	2·492	13·074	·216	78	2·926	·229	2·383	·157
36	47·018	2·390	12·858	·222	79	2·697	·213	2·226	·150
37	44·628	2·290	12·636	·228	80	2·484	·197	2·076	·142
38	42·338	2·195	12·408	·233	81	2·287	·183	1·934	·134
39	40·143	2·103	12·175	·239	82	2·104	·169	1·800	·127
40	38·040	2·012	11·936	·244	83	1·935	·156	1·673	·120
41	36·028	1·927	11·692	·249	84	1·779	·144	1·553	·112
42	34·101	1·843	11·443	·254	85	1·635	·134	1·441	·106
43	32·258	1·762	11·189	·259	86	1·501	·123	1·335	·099
44	30·496	1·685	10·930	·263	87	1·378	·113	1·236	·093
45	28·811	1·609	10·667	·267	88	1·265	·105	1·143	·087
46	27·202	1·537	10·400	·271	89	1·160	·096	1·056	·081
47	25·665	1·466	10·129	·273	90	1·064	·089	·975	·076
48	24·199	1·398	9·856	·277	91	·975	·081	·899	·070
49	22·801	1·332	9·579	·280	92	·894	·075	·829	·066
50	21·469	1·269	9·299	·282	93	·819	·069	·763	·062
51	20·200	1·208	9·017	·283	94	·750	·064	·701	·057
52	18·992	1·148	8·734	·284	95	·686	·058	·644	·053
53	17·844	1·092	8·450	·285	96	·628	·053	·591	·049
54	16·752	1·037	8·165	·286	97	·575	·049	·542	·047
55	15·715	·983	7·879	·285	98	·526	·044	·495	·043
56	14·732	·933	7·594	·284	99	·482	·041	·452	·040
57	13·799	·884	7·310	·283	100	·441	...	·412	...

Carlisle Table—3 *per-cent.*

x	f_{xxxx}	Δ	$(\Delta)^{-1}$	x	f_{xxxx}	Δ	$(\Delta)^{-1}$
15	·000041	·000004	250	58	·002105	·000201	498
16	·000045	·000005	200	59	·002306	·000220	455
17	·000050	·000005	200	60	·002526	·000242	413
18	·000055	·000005	200	61	·002768	·000265	377
19	·000060	·000005	200	62	·003033	·000290	345
20	·000065	·000007	143	63	·003323	·000318	314
21	·000072	·000007	143	64	·003641	·000348	287
22	·000079	·000007	143	65	·003989	·000381	262
23	·000086	·000008	125	66	·004370	·000418	239
24	·000094	·000009	111	67	·004788	·000458	218
25	·000103	·000010	100	68	·005246	·000502	199
26	·000113	·000011	910	69	·005748	·000550	182
27	·000124	·000012	833	70	·006298	·000602	166
28	·000136	·000013	769	71	·006900	·000660	152
29	·000149	·000014	714	72	·007560	·000723	138
30	·000163	·000016	625	73	·008283	·000792	126
31	·000179	·000017	588	74	·009075	·000868	115
32	·000196	·000019	526	75	·009943	·000951	105
33	·000215	·000020	500	76	·010894	·001042	960
34	·000235	·000022	455	77	·011936	·001141	876
35	·000257	·000025	400	78	·013077	·001251	799
36	·000282	·000027	370	79	·014328	·001370	730
37	·000309	·000030	333	80	·015698	·001501	666
38	·000339	·000032	313	81	·017199	·001645	608
39	·000371	·000036	278	82	·018844	·001802	555
40	·000407	·000039	256	83	·020646	·001975	506
41	·000446	·000042	238	84	·022621	·002164	462
42	·060488	·000047	213	85	·024785	·002370	422
43	·000535	·000051	196	86	·027155	·002597	385
44	·000586	·000056	179	87	·029752	·002845	351
45	·000642	·000061	164	88	·032597	·003118	321
46	·000703	·000068	147	89	·035715	·003416	293
47	·000771	·000073	137	90	·039131	·003742	267
48	·000844	·000081	123	91	·042873	·004100	244
49	·000925	·000088	114	92	·046973	·004493	223
50	·001013	·000097	103	93	·051466	·004922	203
51	·001110	·000107	935	94	·056388	·005392	185
52	·001217	·000116	862	95	·061780	·005909	169
53	·001333	·000127	787	96	·067689	·006474	154
54	·001460	·000140	714	97	·074163	·007092	141
55	·001600	·000153	654	98	·081255	·007772	129
56	·001753	·000168	595	99	·089027	·008514	117
57	·001921	·000184	543	100	·097541	..	...

$k_4 = \cdot 13185$

CARLISLE TABLE—3 *per-cent.*

Age	$\bar{a}'_{xxxx}$	$-\Delta\bar{a}'_{xxxx}$	$\bar{a}_{xxxx}$	$-\Delta\bar{a}_{xxxx}$	Age	$\bar{a}'_{xxxx}$	$-\Delta\bar{a}'_{xxxx}$	$\bar{a}_{xxxx}$	$-\Delta\bar{a}_{xxxx}$
15	86·640	3·338	14·247	·093	58	9·873	·637	5·910	·251
16	83·302	3·226	14·154	·097	59	9·236	·603	5·659	·249
17	80·076	3·140	14·057	·102	60	8·633	·571	5·410	·245
18	76·936	3·035	13·955	·107	61	8·062	·540	5·165	·241
19	73·901	2·947	13·848	·111	62	7·522	·510	4·924	·237
20	70·954	2·864	13·737	·116	63	7·012	·481	4·687	·232
21	68·090	2·769	13·621	·121	64	6·531	·453	4·455	·226
22	65·321	2·686	13·500	·126	65	6·078	·426	4·229	·221
23	62·635	2·596	13·374	·131	66	5·652	·401	4·008	·215
24	60·039	2·517	13·243	·136	67	5·251	·376	3·793	·208
25	57·522	2·435	13·107	·142	68	4·875	·353	3·585	·202
26	55·087	2·359	12·965	·148	69	4·522	·331	3·383	·196
27	52·728	2·284	12·817	·153	70	4·191	·310	3·187	·188
28	50·444	2·210	12·664	·158	71	3·881	·289	2·999	·181
29	48·234	2·136	12·506	·164	72	3·592	·270	2·818	·173
30	46·098	2·063	12·342	·170	73	3·322	·253	2·645	·166
31	44·035	1·994	12·172	·176	74	3·069	·235	2·479	·159
32	42·041	1·927	11·996	·181	75	2·834	·218	2·320	·152
33	40·114	1·861	11·815	·187	76	2·616	·204	2·168	·144
34	38·253	1·798	11·628	·193	77	2·412	·189	2·024	·137
35	36·455	1·733	11·435	·198	78	2·223	·175	1·887	·130
36	34·722	1·673	11·237	·204	79	2·048	·163	1·757	·122
37	33·049	1·612	11·033	·209	80	1·885	·150	1·635	·116
38	31·437	1·553	10·824	·214	81	1·735	·140	1·519	·109
39	29·884	1·496	10·610	·220	82	1·595	·129	1·410	·103
40	28·388	1·439	10·390	·224	83	1·466	·119	1·307	·097
41	26·949	1·385	10·166	·229	84	1·347	·110	1·210	·090
42	25·564	1·331	9·937	·234	85	1·237	·102	1·120	·085
43	24·233	1·278	9·703	·238	86	1·135	·093	1·035	·079
44	22·955	1·228	9·465	·241	87	1·042	·086	·956	·074
45	21·727	1·178	9·224	·245	88	·956	·080	·882	·069
46	20·549	1·129	8·979	·249	89	·876	·073	·813	·064
47	19·420	1·082	8·730	·251	90	·803	·067	·749	·060
48	18·338	1·036	8·479	·254	91	·736	·062	·689	·056
49	17·302	·989	8·225	·255	92	·674	·057	·633	·052
50	16·313	·947	7·970	·257	93	·617	·052	·581	·049
51	15·366	·905	7·713	·259	94	·565	·048	·532	·046
52	14·461	·861	7·454	·259	95	·517	·043	·486	·042
53	13·600	·822	7·195	·258	96	·474	·041	·444	·041
54	12·778	·782	6·937	·259	97	·433	·037	·403	·038
55	11·996	·744	6·678	·257	98	·396	·033	·365	·037
56	11·252	·708	6·421	·256	99	·363	·031	.328	·030
57	10·544	·671	6·165	·255	100	·332	...	·298	...

Mr. T. G. C. Browne's Method of Division of Profits.

[Extracted from *The Insurance Record* of 19 July 1895].

(To the Editor of "The Insurance Record.")

Sir,—In his letter, which appeared in your issue of 26 April last, dealing with Mr. Lidstone's paper read before the Institute and the discussion thereon, Dr. Sprague invited me to give some figures illustrating my method of division, but I delayed doing so until I was in a position to publish the figures of my office up to date. I now beg to send you the analysis of the net profit actually divided among the policyholders during the 20 years 1875-1894 inclusive, and I venture to think that it is of some general interest from the principles it illustrates. The profit is divided into three groups, as described in the following statement, and Group I increased or decreased by its rateable proportion of Group III is divided according to the loading received during the five years; while Group II similarly increased or decreased, is divided according to the reserve value at the *end* of that period. This method of division was applied consistently to all classes of policies entitled to participate. In the following statement the various items of profit included in Group III are given separately, as it is upon the mode of dealing with them that discussion is most likely to arise. The sign + is prefixed to a profit and — to a loss:—

[*See next page.*]

In looking at the figures for 1875-1879, at the end of which period my method was first adopted, the prominent feature is the very large loss from mortality, and it was apparent that neither Dr. Sprague's method nor that of the Australian Mutual Provident could be used satisfactorily under the circumstances. I had therefore to consider what modifications were necessary to make the latter (which I preferred) applicable to the case before me. The creation of a third group to be rateably divided between the other two, appeared to me absolutely necessary in order to make a contribution method work satisfactorily in times of adversity as well as of prosperity, and I had to decide how that group should be composed. The peculiar circumstances of the quinquennium may have influenced me in the choice of the items to be included in it, but I have seen no reason to alter it during the succeeding three periods, although a modification, which I shall refer to presently, may be desirable hereafter.

When the object of a third group is kept in view, the principle of its composition is obviously that it should include all important items which are likely to fluctuate greatly from period to period, thus disturbing the practical working of the system. The various items of Group III, I think, fairly comply with these conditions, except one—"Bonuses on Re-assurances," which I included simply for the sake of convenience. It represents the present cash value of the total bonuses on re-assurances declared during the quinquennium,

Analysis of Profits.

	1875-9.	1880-4.	1885-9.	1890-4.
GROUP I:				
Loading, less Expenses and Commission, Surrenders, and Lapses	+ 36,454	+ 49,143	+ 62,719	+ 63,587
GROUP II:				
Interest earned above 3 per-cent, including Interest earned on Reversions at an assumed rate	+ 81,941	+ 88,986	+ 85,907	+ 79,144
GROUP III:				
(1) Mortality	− 29,282	+ 6,411	+ 47,306	+ 48,620
(2) Investments realized, including the profit from Reversions, less the assumed rate of Interest included in Group II .	+ 6,487	+ 7,414	− 1,474	+ 10,811
(3) Bonuses on Re-assurances	+ 4,596	+ 7,278	+ 13,912	+ 12,784
(4) Interest on Investment Reserve Fund, undivided Balance from preceding period, and on accrued profit	+ 11,804	+ 14,283	+ 15,762	+ 22,188
(5) Issue Policies . . .	...	+ 1,413	+ 3,802	+ 2,560
(6) Annuities*	...	...	− 1,511	+ 3,436
Total of Group III .	− 6,395	+ 36,799	+ 77,797	+100,399
Total Amount divided among Policyholders in force at end of period	+112,000	+175,228	+226,423	+243,130
Loading Multiplier	·461	·525	·740	·772
Reserve „	·0733	·0764	·0773	·0736

NOTE.—The business of the London and Provincial Law Office was taken over in 1882—hence the large increase in the figures dealt with.

* These were kept until recent years in a separate fund.

and strictly it is not a profit at all, except so far as it exceeds the bonuses declared on the re-assured portion of the principal policies. To bring these bonuses into account year by year as they are declared and to set them against the principal policies, would involve a considerable amount of trouble, and they have therefore been dealt with in a lump sum at the end of the quinquennium as an item in Group III. Dr. Sprague states that he is opposed to my views as to the treatment of profit or loss from investments, and he argues that this item should be treated as an addition to or deduction from interest. Theoretically I agree with him, and if such an item

occurred with regularity in one direction only, there would be no objection to treating it as he suggests, but unfortunately an examination of the accounts of many companies shows that this item is perhaps the most variable that the actuary has to deal with; I contend therefore that if there is a necessity for a third group at all it should be included in it. I take a similar view of the profit from reversions above an assumed rate of interest.

In the foregoing statement, I give the Loading and Reserve Multipliers for the four periods; thus, for 1880-84 these are ·525 and ·0764 respectively, that is to say, a little more than one-half of the loading and upwards of $1\frac{1}{2}$ per-cent per annum on the reserve was apportioned in cash to the policyholders, who received the full cash apportionment without any deduction, or a corresponding reversion on an H^{M} or $H^{M(5)}$ 3 per-cent basis.

The following specimens of the reversionary bonuses declared at the four divisions during which the profits earned have varied greatly, is the best test of my method as a practical one:

Age at Entry	Year	Number of Years in Force																	
		5		10		15		20		25		30		35		40		45	
		£	s.	£	s.	£	s.	£	s.	£	s.	£	s.	£	s.	£	s.	£	s.
30	1879	4	8	5	6	6	2	6	18	7	14	8	0	8	13	9	6	10	0
	1884	5	6	6	4	7	4	8	2	8	18	9	14	10	0	10	14	11	6
	1889	6	16	7	14	8	12	9	10	10	6	11	0	11	14	11	18	12	10
	1894	*6	2	7	18	8	14	9	12	10	10	11	4	11	18	12	12	12	14
40	1879	4	1	5	2	6	2	7	1	7	19	8	8	9	3	10	0	10	19
	1884	4	15	5	17	7	0	7	19	8	18	9	16	10	2	10	19	11	15
	1889	6	2	7	4	8	6	9	8	10	8	11	4	12	2	12	8	13	4
	1894	6	2	7	6	8	6	9	6	10	8	11	4	12	2	12	18	13	4

* The decline in this bonus as compared with 1889 is accounted for by a reduction in the rate of premium at age 30.

At the end of the last quinquennium, in consequence of the continuous fall in the rate of interest, it was decided to carry forward a much larger balance of undivided profit than on previous occasions instead of lowering the valuation rate of interest. Under these circumstances, I think that in future the interest on this balance might be included in Group II, together with interest on any investment reserve fund which may exist, provided that such fund has been created purely as a general precautionary measure and not to meet specific losses actually in view.

Dr. Sprague points to the fact that his old office has adhered to his method since he introduced it in 1865, and he gives specimens of the bonuses up to 1890, but he admits that the results have been effected by the exceptional profits earned. Since he wrote his letter the results for 1895 have been published, and they may possibly have altered his opinion of his system, but whether this be so or not, I

venture to predict that, before many years are over, his old office will be obliged to modify it.

I am, Sir, your obedient Servant,

T. G. C. BROWNE.

6 Princes Street, E.C.
16 *July* 1895.

CORRESPONDENCE.

"MAXIMUM MORTALITY PERCENTAGES."

To the Editor of the Journal of the Institute of Actuaries.

Sir,—I have carefully read Mr. T. B. Macaulay's letter (*J.I.A.*, xxxii, 117) with the above heading, and I consider it an interesting contribution to the discussion of the important subject of the mortality among insured lives.

In my paper of 1870, I showed, from an examination of the experience collected by the Institute of Actuaries, that, taking all ages together, and comparing the actual deaths with the expected, in suitable groups of years of insurance, the series of ratios exhibited a maximum for the group of years of insurance, 11 to 15. Mr. Macaulay has now conclusively shown that such a maximum may appear in an aggregate for all ages, even when it does not appear in any of the series obtained from the same statistics arranged in quinquennial groups of ages.

I think, however, that Mr. Macaulay attaches too much importance to his results, and that they do not bear out the conclusions he draws from them. I would first point out that it is very incorrect to speak, as he does, of a maximum mortality "theory"; for the question whether there is a maximum in the series of percentages, or not, is a question of fact, and not of theory. So, again, when he speaks of my "opinion" on this point, I have to remind him that this is not a matter of opinion, but a matter of fact. Either there is a maximum, or there is not; any man's mere opinion on such a point is of very little weight, if any; and the question must be decided by an examination of the statistics.

What, then, are the facts? The table given by Mr. Macaulay enables us to answer this question very readily, although if we wish to have the years of insurance correctly, we must refer to the *J.I.A.* for April 1870. My description of the facts was as follows (page 349), "The law * * * of the actual mortality as compared with "the expected, increasing to a maximum, and then again decreasing, "is retained in a very marked manner throughout the interval of 20 "years, from age 56 to 75; and it will be seen further that the "figures involved are so large, and the results are so consistent with "each other, that we are forced to admit the phenomenon as a true "deduction from the facts. At both extremities of the above period, "namely from the age 46 to 55 at the one end, and from 76 to 80 at

" the other end, the same law appears in a less prominent manner; " while under the age of 46, and above the age of 80, the rate of " mortality increases continually with the standing of the assurance." In order to show more clearly the nature of the facts which Mr. Macaulay endeavours to explain away, I have extracted from my paper the figures in columns (2) to (7) of the following table, and added those in the last column.

Present Ages (1)	Number at Risk (2)	Deaths (3)	Maximum Percentage (4)	Years of Insurance (5)	Final Percentage (6)	Years of Insurance (7)	Difference (4)−(6) (8)
46–50	156,180·5	2,255	109·7	19–25	106·8	26–35	2·9
51–55	124,028·5	2,302	113·6	19–22	107·7	23–40	5·9
56–60	90,757	2,312	108·7	12–17	93·6	30–end	15·1
61–65	60,778·5	2,258	106·4	16–21	100·0	32– „	6·4
66–70	36,283	1,920	105·8	14–16	99·7	27– „	6·1
71–75	19,012·5	1,548	105·2	18–21	90·7	33– „	14·5
76–80	7,808·5	937	104·4	27–36	103·1	37– „	1·3
	494,848·5	13,532					

These facts are not to be got rid of by arguments such as those used by Mr. Macaulay. If the law indicated by the figures, namely the increase of the percentages to a maximum, and their subsequent decrease, were not a reality; but, as Mr. Macaulay supposes, the result of calculating the expected deaths by an unsuitable table; this could be proved by calculating the expected deaths by a more suitable table: and I trust that Mr. Macaulay will apply this test to the figures, and publish the results. I am satisfied that, when this is done, it will be found that the same law still prevails in the series of percentages. It is certainly true, as pointed out by me in 1870, and now in greater detail by Mr. Macaulay, that the use of an incorrect mortality table may, when the expected deaths *at all ages* are compared with the actual, introduce a maximum into the series of percentages, which would not be there if a correct mortality table were used. But, when this comparison is restricted to a small interval of age, such as five years, inaccuracies in the mortality table, even of the most serious character, can produce no such effect. If, for instance, the mortality table employed gave, in a certain quinquennial group of ages, a mortality 50 per-cent in excess of the truth, *all* the percentages relating to persons who had attained those ages, but had been insured for different periods, would be reduced in the same ratio; each of them being two-thirds of what it would be if a true table had been used; but the inaccuracy in the table would not introduce any maximum into the series of percentages.

The law of the figures being such as I have described, it is a wholly different question how this law is to be explained, or accounted for. In 1870, I showed how the law might be accounted for as the effect of the withdrawal of healthy lives; and I believed

that its existence proved that the lives which withdraw are, on the average, better than those which remain. I now entertain considerable doubt on this point. Mr. Chatham claims to have shown (*J.I.A.*, xxix, 81), that, where the withdrawals are most numerous in the early years of insurance, there the mortality is lightest (p. 172); and he concludes, "that in ordinary circumstances the rate of increase in "the mortality during the 10 years after insurance is independent of "the rate of discontinuance": (p. 173). One of the facts on which he relies is that, on comparing the experience of the 10 Scotch offices with the total experience collected by the Institute in 1863, the rate of withdrawal in the Scotch experience was less than in the total, of which it formed a part; but the rate of mortality was, nevertheless, higher. Mr. Chatham's conclusion, if correct, is of great importance; but the difference between the rates of mortality in the Scotch and in the total experience, is small, and the conclusion seems to me to require confirmation from other sources, before it can be accepted without reserve.

The results of a careful examination which I have made of the mortality among the Female Government Annuitants, as given in Mr. A. J. Finlaison's Report of 1882, seem to point in the same direction. I hope to submit the details of my investigation to the Institute at some future time, but I take the present opportunity of stating the general results. I must, however, do this from memory, as I have mislaid my notes on the subject. On treating the annuitant experience in the same way as I treated the experience of assured lives in 1870, I found a very similar law prevailed in the results; that is to say, taking quinquennial groups of ages attained, I found that, in almost all of these groups, as the time elapst since the date of purchase continually increast, the ratio of the actual to the expected claims, first increast to a maximum, and then diminisht. In this case, the law cannot be a result of withdrawals, for there are none; and some other explanation of the law must be sought for. The only possible explanation seems to be a peculiar composition of the body of lives on which annuities are purchast. Suppose, for example, that the nominees of any age on whose lives annuities are purchast, comprise (1) a majority, who are subject to a low rate of mortality at first, gradually increasing in the course of, say, 10 years, to the normal rate corresponding to the age; and (2) a minority, who are also subject to a low rate of mortality at first, but, after the lapse of a few years, are subject to such heavy rates of mortality that they are all dead within, say, 25 years from the date of purchase; then, confining our attention to nominees of a given age at entry, it is clear that during the 10 years after purchase, the ratio of the actual to the expected deaths, as calculated according to the normal rate of mortality, will continually increase: that, after the lapse of the 10 years, it will still increase, in consequence of the heavy mortality among the minority lives, and will for several years exceed unity; but, that, after the lapse of the 25 years, when all the minority lives are dead, the ratio will become equal to unity; in other words, the ratio will attain a maximum and then diminish. This is, of course, not the same case as we have been hitherto considering; for we have here traced a number of entrants of

the same age, say 50, who have gradually grown older; but it is not difficult to see that, under the circumstances supposed, the same law will hold good when we consider nominees of different ages at entry, who attain the same age, say 70, after the lapse of different times.

Further investigation is desirable, in order to ascertain, if possible, whether the composition of the body of annuitant lives is such as I have described above; or, at all events, whether the law of the percentages increasing to a maximum and then diminishing is observed in other collections of data. If so, it may fairly be argued that the same cause which produces this maximum among annuitant lives, produces it, or at all events, contributes to it, in the case of assured lives.

I am, Sir,

Your obedient servant,

T. B. SPRAGUE.

Edinburgh,
28 *August* 1895.

The Canadian Insurance Acts, 1894 *and* 1895.

THE INSURANCE ACT, CHAP. 124, REVISED STATUTES OF CANADA (1886), AS AMENDED BY 51 VIC., CHAP. 28 (1888), AND 57 VIC., CHAP. 20 (1894).

[The Act of 1888, assented to 22 May of that year, amended sub-section (*c*) of section 3; all the other amendments were made by the Act of 1894, assented to 23 July of the year last mentioned.]

Preamble. HER Majesty, by and with advice and consent of the Senate and House of Commons of Canada, enacts as follows:

Short Title.

Short title. 1. This Act may be cited as "The Insurance Act."

Interpretation.

Interpretation. 2. In this Act, unless the context otherwise requires:

"Minister." (*a*) The expression "Minister" means the Minister of Finance and Receiver General;

"Superintendent." (*b*) The expression "Superintendent" means the Superintendent of Insurance;

"Company." (*c*) The expression "company" means and includes any corporation or any society or association, incorporated or unincorporated, or any partnership carrying on the business of insurance;

"Canadian company." (*d*) The expression "Canadian company" means a company incorporated or legally formed in Canada, for the purpose of carrying on the business of insurance in Canada, and which has its head office therein;

(*e*) The expression "agent" means the chief agent of the company in Canada, named as such in the power of attorney hereinafter referred to, by whatever name he is designated; "Agent."

(*f*) The expression "chief agency" means the principal office or place of business of the company in Canada: "Chief Agency."

(*g*) The expression "inland marine insurance" means marine insurance in respect to subjects of insurance at risk upon the waters of Canada above the harbour of Montreal; "Inland Marine Insurance."

(*h*) The expression "Canadian policy" or "policy in Canada", as regards life insurance, means a policy issued by any company licensed under this Act to transact the business of life insurance in Canada, in favour of any person or persons resident in Canada at the time when such policy was issued: and "policyholder in Canada" means any such person as aforesaid; "Canadian policy."

(*i*) The expression "Canadian policy" or "policy in Canada", as regards fire and inland marine insurance, means a policy of insurance on any property within Canada, issued by any company licensed under this Act to transact the business of fire or inland marine insurance; The same.

(*j*) The expression "license" includes certificate of registration;

(*k*) The expression "policy" includes a certificate of membership relating in any way to life insurance, and any other written contract of insurance whether contained in one or more documents.

Application of Act.

3. The provisions of this Act shall not apply— To what companies this Act shall not apply.

(*a*) To any company transacting, in Canada, ocean marine insurance exclusively; or— Marine.

(*b*) To any policy of life insurance in Canada, issued previously to the twenty-second day of May, in the year one thousand eight hundred and sixty-eight, by any company which has not subsequently received a license; or— Policies prior to 22 May 1868.

(*c*) To any company incorporated by an Act of the Legislature of the late province of Canada, or by an Act of the Legislature of any province now forming part of Canada, which carries on the business of insurance, wholly within the limits of that province by the Legislature of which it was incorporated, and which is within the exclusive control of the Legislature of such province; but any such company may, by leave of the Governor in Council, on complying with the provisions of this Act, avail itself of the provisions of this Act, and if it so avails itself, the provisions of this Act Certain companies under Provincial Acts. Proviso.

shall thereafter apply to it, and such company shall have the power of transacting its business of insurance throughout Canada.

Licenses.

What companies or persons only may transact life, fire or inland marine insurance business in Canada.

4. No company or person, except as hereinafter provided, shall accept any risk or issue any policy of fire or inland marine insurance or policy of life insurance, or grant any annuity on a life or lives, or receive any premium, or carry on any business of life or fire or inland marine insurance in Canada,—or prosecute or maintain any suit, action or proceeding, either at law or in equity, or file any claim in insolvency relating to such business, without first obtaining a license from the Minister to carry on such business in Canada.

Form and duration of license.

5. The license shall be in such form as is, from time to time, determined by the Minister, and shall specify the business to be carried on by the company; and it shall expire on the thirty-first day of March in each year, but shall be renewable from year to year.

License to issue on certain conditions only.

6. The Minister, as soon as the company applying for the same has deposited in his hands the securities hereinafter mentioned, and has otherwise conformed to the requirements of this Act, shall issue such license as aforesaid.

Enumeration of classes of insurance business that may or may not be done together.

6a. A license shall not be granted to a company to carry on the business of life insurance in combination with any other branch of insurance.

2. A license may be granted to a company to carry on the three following classes of insurance, viz., fire insurance, inland marine insurance and cyclone or tornado insurance or any two of the said classes.

3. A license may be granted to a company to carry on any two of the following classes of insurance:

(*a*) Guarantee insurance, that is to say: to guarantee the fidelity of persons in positions of trust;

(*b*) Accident insurance, that is to say: to insure against bodily injury and death by accident, including the liability of employers for injuries to persons in their employment;

(*c*) Plate-glass insurance, that is to say: to insure against the breakage of plate or other glass either local or in transit;

(*d*) Steam boiler insurance, that is to say: to insure against loss or damage to the life, person or property of the insured, or of another for which the insured is liable, caused by the explosion of steam boilers.

As to unenumerated classes of insurance business that may be done together.

4. A license may, on the recommendation of the Superintendent of Insurance approved by the Treasury Board, be granted to a company to carry on any class or classes of insurance not hereinbefore enumerated, but no such license shall be granted for more than two such

classes of insurance, or on the like recommendation approved as aforesaid, a license may be granted to a company to carry on one of the classes of business above enumerated, and one other class of insurance not above enumerated.

5. Except as hereinbefore provided, a license shall not be granted to a company to carry on more than two classes of business. Limitation.

(The amending Act provides that this section shall not interfere with the renewal of licenses granted before the passing of the Act, 23 July 1894, or with any application for license pending on the first day of April 1894.)

Provision for the case of companies having power by law to carry on more classes of business than allowed under sec. 6A.

6B. A license shall not be granted to a company which is by its charter authorized or empowered to carry on classes or branches of insurance greater in number or variety than those for which a license could be granted under the provisions of the next preceding section. Provided, however, that any company incorporated elsewhere than in Canada, regardless of its charter powers, which has a paid-up capital, in the case of a company authorized to transact among other classes of business the business of fire insurance, of at least three hundred thousand dollars, and in the case of any other company, of at least one hundred thousand dollars, wholly unimpaired and in addition to such paid-up capital holds over and above all liabilities estimated according to the existing Dominion Government standard, a rest or surplus fund equal to at least twenty per-cent of such paid-up capital, and the market value of whose stock is at a premium of at least twenty per-cent, and which has carried on successfully for a period of at least five years, the business for which a license is sought, being only one class of insurance, or if more than one then such classes as may be combined under the provisions of the next preceding section shall be deemed eligible for and entitled to such license upon depositing, keeping and maintaining assets in Canada as defined by sub-sections two and three of section ten of this Act, over and above and in excess of the amount which would be required, if such company's charter powers were limited to the purposes for which such license is asked, to such an amount as the Treasury Board, on the report of the Superintendent, shall fix or determine, not being less than ten thousand dollars nor more than two hundred thousand dollars, such excess to be regarded as the company's Canadian capital.

(The amending Act provides that this section shall not interfere with the renewal of licenses granted before the passing of the Act, 23 July 1894.)

DEPOSITS TO BE MADE BEFORE THE ISSUE OF LICENSE.

Deposit of securities with the Minister, and to what amount.

7. Every company carrying on the business of life insurance, and every Canadian company carrying on the business of fire or of inland marine insurance, or of both combined, shall, before the issue of such license, deposit with the Minister, in such securities as are hereinafter mentioned, the sum of fifty thousand dollars; and every company incorporated or legally formed out of Canada, carrying on the business of fire or of inland marine insurance, or of both combined, shall, before the issue of such license, deposit with the Minister, in such securities as are hereinafter mentioned, the sum of one hundred thousand dollars.

Nature of securities that may be deposited.

8. All such deposits may be made by any company in securities of the Dominion of Canada, or in securities issued by any of the provinces of Canada; and by any company incorporated in the United Kingdom, in securities of the United Kingdom: and by any company incorporated in the United States, in securities of the United States;

Valuation.

and the value of such securities shall be estimated at their market value, not exceeding par, at the time when they are so deposited:

Valuation of other securities.

2. If any securities other than those above mentioned are offered as a deposit, they may be accepted, at such valuation and on such conditions as the Treasury Board directs:

If the value declines, further deposit.

3. If the market value of any of the securities which have been deposited by any company declines below that at which they were deposited, the Minister may notify the company to make a further deposit, so that the market value of all the securities deposited by the company shall be equal to the amount which it is

Penalty for failure.

required by this Act to deposit; and on failure by the company to make such further deposit within sixty days after being called upon so to do, the Minister may withdraw its license:

Company may deposit further security.

4. Any company licensed under this Act may, at any time, deposit in the hands of the Minister any further sums of money or securities beyond the sum herein required to be deposited; and any such further sums of money, or securities therefor, so deposited in the hands of

How to be dealt with.

the Minister, shall be held by him and be dealt with according to the provisions of this Act in respect to the sum required to be deposited by such company, and as if the same had been part of the sum so required to be deposited:

Treasury Board may authorize withdrawal of deposit in excess.

5. If at any time it appears that a company has on deposit with the Minister a sum in excess of the amount required under the provisions of this Act, the Treasury Board may, upon being satisfied that the interest of the company's Canadian policyholders will not be prejudiced

thereby, and upon the giving of such notice, and the exercise of such other precautions as may seem expedient, authorize the withdrawal of the amount of such excess or such portion thereof as may be deemed advisable: provided that such withdrawal may be authorized without the giving of any notice.

9. If it appears from the annual statements, or from an examination of the affairs and condition of any company carrying on the business of fire or inland marine insurance, that the re-insurance value of all its risks outstanding in Canada, together with other liabilities in Canada, exceeds its assets in Canada, including the deposit in the hands of the Minister, the company shall be notified by the Minister to make good the deficiency; and on its failure so to do, within sixty days after being so notified, he shall withdraw its license.

Any deficiency of security to be made good by fire and inland marine companies.

Penalty for default.

10. If it appears from the annual statements, or from an examination, as provided for by this Act, of the affairs and conditions of any company carrying on the business of life insurance, that its liabilities to policyholders in Canada, including matured claims, and the full reserve or re-insurance value for outstanding policies, as hereinafter described, after deducting any claim the company has against such policies, exceed its assets in Canada, including the deposit in the hands of the Minister, the company shall be called upon by the Minister to make good the deficiency; and on its failure so to do within sixty days, he shall withdraw its license:

Any deficiency of security by other companies to be made good by them.

Penalty for default.

2. If any such company as is mentioned in this and the next preceding section is incorporated or legally formed elsewhere than within Canada, the assets in Canada as aforesaid shall be taken to consist of all deposits which the company has made with the Minister under the foregoing provisions of this Act, and of such assets as have been vested in trust for the Company for the purposes of this Act, in two or more persons resident in Canada, appointed by the company and approved by the Minister:

As to company incorporated elsewhere than in Canada.

Deposits and assets to be vested in trust.

3. The trust deed shall first be approved of by the Minister, and the trustees may deal with such assets in any manner provided by the deed of trust appointing them, but so that the value of the assets held by them shall not fall below the value required by this section:

Trust deed and dealing with assets.

4. In the case of any such life insurance company, which gave written notice to the Minister before the thirty-first day of March, in the year one thousand eight hundred and seventy-eight, of its intention to avail itself of the proviso contained in section seven of "The Consolidated Insurance Act, 1877", the foregoing requirements of this section shall not apply to policies issued previously to that date; and the deposit of such

Provision as to companies which gave notice before 31 March 1878.

company, which was in the hands of the Minister, on the twenty-eighth day of April, in the year one thousand eight hundred and seventy-seven, shall be dealt with in regard to such policies, in conformity with the fourth and fifth sections of an Act passed by the Parliament of Canada in the thirty-fourth year of Her Majesty's reign, intituled "An Act to amend the Act respecting Insurance Companies"; and whenever the full liability under such policies falls below the amount so held by the Minister, he may, with the concurrence of the Treasury Board, direct that the whole or such portion of the difference as he deems advisable, shall be released and handed over to the company, and so on, from time to time, until the total deposit with the Minister is reduced to the amount of fifty thousand dollars required by this Act.

Release of surplus securities to such company.

Interest on securities. When to be payable to company.

11. So long as the conditions of this Act are satisfied by any company, and no notice of any final judgment against the company, or order made by the proper court in that behalf for the winding up of the company or the distribution of its assets, is served upon the Minister, the interest upon the securities forming the deposit shall be handed over to the company as it falls due.

Documents to be Filed.

Certain documents to be filed.

12. Every company shall, before the issue of a license to it, file in the Department of Finance,—

Copy of charter.

(*a*) A copy of the charter, Act of incorporation, or articles of association of the company, certified by the proper officer in charge of the original thereof;

Power of attorney to agent in Canada.

(*b*) A power of attorney from the company to its agent in Canada, under the seal of the company, if it has a seal, and signed by the president and secretary or other proper officers thereof, in presence of a witness, who shall make oath or affirmation as to the due execution thereof; and the official positions in the company held by the officers signing such power of attorney shall be sworn to or affirmed by some person cognizant of the facts necessary in that behalf; and—

Statement of affairs of company.

(*c*) A statement in such form as is required by the Minister, of the condition and affairs of such company on the thirty-first day of December then next preceding, or up to the usual balancing day of the company, if such day is not more than twelve months before the filing of the statement.

What the power of attorney must contain.

13. Such power of attorney shall declare at what place in Canada the head office, or chief agency of such company is, or is to be established,—and shall expressly authorize such attorney to receive service of process in all suits and

proceedings against such company in any province of Canada, in respect of any liabilities incurred by the company therein, and also to receive from the Minister and the Superintendent, all notices which the law requires to be given, or which it is thought advisable to give,—and shall declare that service of process for or in respect of such liabilities, and receipt of such notices, at such office or chief agency, or personally on or by such attorney at the place where such head office or chief agency is established, shall be legal and binding on the company to all intents and purposes whatsoever.

Provision to be made for service of process.

14. Whenever any such company changes its chief agent or chief agency in Canada, such company shall file a power of attorney, as hereinbefore mentioned, containing any such change or changes in such respect, and containing a similar declaration as to service of process and notices as hereinbefore mentioned; and every company shall at the time of making the annual statement hereinafter provided for, declare that no change or amendment has been made in the charter, Act of incorporation or articles of association of the company, and that no change has been made in the chief agency or chief agent, without such amendment or change having been duly notified to the Superintendent.

If changes are made in a chief agency.

Declaration of no change to be made in charter, &c.

15. Duplicates of all such documents, duly verified as aforesaid, shall be filed in the office of one of the superior courts in the province in which the head office or chief agency of the company is situated,—or if the chief agency is in the province of Quebec, with the prothonotary of the Superior Court of the district wherein such chief agency is established.

Duplicates of such documents to be filed in court.

SERVICES OF COMPANIES WITH PROCESS.

16. After such power of attorney and certified copies are filed as aforesaid, any process in any suit or proceeding against any such company, in respect of any liabilities incurred in any province of Canada, may be validly served on the company at its chief agency; and such service shall be deemed to be service on the company:

Service of process on company.

2. If such power of attorney becomes invalid or ineffective from any reason whatsoever, or if other service cannot be effected, the court or a judge may order constructive service of any process or proceeding to be made by such publication as is deemed requisite to be made in the premises for at least one month in at least one newspaper; and such publication shall be held to be due service upon the company of such process or proceeding.

Constructive service of process.

Notice of License.

Company to give notice of license.

17. Every company on first obtaining such license shall forthwith give due notice thereof in the *Canada Gazette*, and in at least one newspaper in the country, city or place where the head office or chief agency is established, and shall continue the publication thereof for the space of four weeks:

And of ceasing business.

2. The like notice shall be given, for the space of three calendar months, when a company ceases, or gives notice that it intends to cease, to carry on business in Canada, such notice to be a condition precedent to the release of the company's deposit.

Publication of Licensed Companies.

Notices by Minister of companies licensed.

18. The Minister shall cause to be published quarterly in the *Canada Gazette* a list of the companies licensed under this Act, with the amount of deposits made by each company; and upon any new company being licensed, or upon the license of any company being withdrawn in the interval between two such quarterly statements, he shall publish a notice thereof in the *Canada Gazette* for the space of four weeks.

Annual Returns by Companies.

Annual statement to be made on oath.

19. The president, vice-president or managing director, and the secretary, actuary or manager of every Canadian company licensed under this Act, shall prepare annually, under their own oath, a statement of the condition and affairs of such company at the thirty-first day of December in each year, which statement shall exhibit the assets and liabilities of the company, and its income and expenditure during the previous year, and such other information as is deemed necessary by the Minister.

Contents.

Form for Life Insurance.

2. In the case of such companies carrying on the business of life insurance, such statement shall be made in the form and manner set forth in the Form A in the schedule to this Act; suitable changes being made therein in the case of companies carrying on business on the assessment plan.

Form for Fire or Inland Marine.

3. In the case of such companies carrying on the business of fire or inland marine insurance, such statement shall be made in the form and manner set forth in the Form B in the schedule to this Act.

Form for other companies.

4. In the case of such companies carrying on business other than life, fire or inland marine insurance, such statement shall be made in the form and manner set forth in the said Form B, as nearly as circumstances will permit, necessary changes only being made therein.

5. Such statements shall be sworn to before some person duly authorized to administer oaths in any legal proceeding, in the Form C in the schedule to this Act.

Before whom to be sworn.

6. The Minister may, from time to time, make such changes in the form of such statements as seem best adapted to elicit from the companies a true exhibit of their condition in respect to the several points hereinbefore enumerated.

Minister may alter form of statement.

20. Every company incorporated or legally formed elsewhere than in Canada, and at present licensed or hereafter licensed under this Act, and every company which is subject to the provisions of this Act, shall make annual statements of its condition and affairs, at the balancing day of the company in the preceding year, and the form and manner of making such statements shall, as to the Canada business of such company, be the same, so far as applicable, as is required of Canadian companies, and as to its general business, shall be in such form as such company is required by law to furnish to the Government of the country in which its head office is situate. The blank forms of the statements of the Canada business shall be supplied by the Superintendent.

Annual statements by companies incorporated elsewhere than in Canada.

Blank forms as to Canada business.

2. Such statements shall, as to the Canada business, be verified by the oath of the company's chief agent, in Canada, and as to the general business, shall be verified by the oath of the president, vice-president or managing director and secretary or actuary of the company.

To be verified on oath and by whom.

3. Such chief agent shall keep at his chief agency in Canada records and documents sufficient to enable him to prepare and furnish the statement of Canada business in this section provided for, and such that the said statement of Canada business may be readily verified therefrom: Provided that in the case of any company having in Canada in addition to such chief agent one or more general agents reporting to the head office, and not to such chief agent, the requirements of this sub-section shall be sufficiently complied with by such chief agent keeping on file at the chief agency, in addition to the necessary records and documents relating to the business transacted by or through such chief agent, annual statements of the business transacted by each such general agent, duly verified by the oath of each such general agent, and such additional records and documents transmitted through the company's head office as shall, taken together, show the company's entire Canadian business.

Set of books, &c., to be kept at chief agency in Canada.

Proviso: as to business done by general agents reporting to Head Office.

4. The statements of the business of general agents in the next preceding sub-section provided for, shall be made up to the thirty-first day of December in each year, and the blank forms for such statements shall, on application, be furnished by the Superintendent.

Date of statements.

Supply of forms.

Construction of proviso to sub-section 3.

5. In the case of any company not availing itself of the proviso contained in sub-section three of this section, such sub-section shall be read and construed without reference to such proviso, and as if the said proviso and the sub-section next preceding this sub-section did not exist.

Validation of past statements.

6. In every case where a company incorporated or legally formed elsewhere than in Canada, has heretofore made and filed with the Minister statements verified under oath, it is hereby declared that such statements and verification were and shall be deemed to have been, and to be sufficient within the intent and meaning of this section.

Date for deposit of statements.

7. The statements mentioned in this and the next preceding section shall be deposited in the office of the Superintendent of Insurance on the first day of January next following the date thereof or within two months thereafter.

Preliminary abstract to be sent.

8. All companies, whether Canadian or otherwise, carrying on the business of life insurance shall, on or before the first day of February in each year, send to the Superintendent a preliminary abstract of the year's

Contents.

Canada business to the preceding thirty-first day of December inclusive. Such abstract shall comprise the cash premium receipts of the year, the number and amount of the policies issued and taken up during the year, the number and amount of policies that are in force at the date of the abstract, the number and amount of the policies that have become claims during the year, and the number and amount of those that have been paid up to the date of the statement, distinguishing as to such as are unpaid between those resisted and unresisted. Such

Verification.

preliminary abstracts shall be verified in the same manner as the annual statements hereinbefore provided for are required to be verified.

Penalty for default.

21. Every company which makes default in depositing in the office of the Superintendent the annual statement hereinbefore provided for, shall incur a penalty of ten dollars for each day during which such default continues;

Recovery of penalties.

all such penalties shall be recoverable and enforceable with costs at the suit of Her Majesty, instituted by the Attorney General of Canada, and shall when recovered be applied towards payment of the expenses of the office of the Superintendent.

Suspension, withdrawal or non-renewal of license if penalty is not paid.

2. If such penalties are not paid, the Minister, with the concurrence of the Treasury Board, may order the license of such company to be suspended or withdrawn as is deemed expedient, and until such penalties are paid, the license of such company shall not on expiry be renewed.

Penalty for issuing any policy in contravention of this Act.

22. Every person who delivers any policy of insurance, or interim receipt, or who collects any premium (except only on policies of life insurance issued to persons not

resident in Canada at the time of issue) or carries on any business of insurance on behalf of any life, fire or inland marine insurance company, without such license as aforesaid, shall, on summary conviction thereof, before any two justices of the peace or any magistrate having the powers of two justices of the peace, for a first offence, incur a penalty of not less than twenty dollars and costs and not more than fifty dollars and costs, and in default of payment the offender shall be liable to imprisonment with or without hard labour for a term of not less than one month nor more than three months; and for a second or any subsequent offence such offender shall be imprisoned with hard labour for a term not less than three months nor more than six months: First offence. Second or subsequent offence.

2. One half of any such penalty when recovered shall belong to Her Majesty and the other half thereof to the informer. Application of penalty.

23. All informations or complaints for the prosecution of offences under the provisions of sections twenty-two, twenty-five and forty-two of this Act shall be laid or made in writing within one year after the commission of the offence. Limitation of time for prosecution.

24. Unless otherwise provided in the special Act incorporating any insurance company, passed by the Parliament of Canada after the twenty-eighth day of April, one thousand eight hundred and seventy-seven, or hereafter to be passed, such special Act and all Acts amending the same shall expire and cease to be in force at the expiration of two years from the passing thereof, unless within such two years the Company thereby incorporated obtains a license from the Minister under the provisions of this Act. Limitation of time of duration of special Acts.

Superintendent and his Duties.

25. The Governor in Council may appoint an officer, to be called the Superintendent of Insurance, who shall act under the instructions of the Minister, and shall examine and report to the Minister, from time to time, upon all matters connected with insurance, as carried on by the several companies licensed to do business in Canada, or required by this Act to make returns of their affairs; Superintendent of insurance: appointment and duties.

2. Such Superintendent may be appointed at a salary not exceeding four thousand dollars per annum; Salary.

3. The Governor in Council may, from time to time, appoint such officers and clerks under the Superintendent, as are necessary for the purposes of this Act: Officers and clerks.

4. The Superintendent shall keep a record of the several documents required to be filed by each company in the superior courts of Canada, under this Act: and shall,— Duties of Superintendent.

Entry of securities deposited.

(*a*) Enter in a book, under the heading of each company, the securities deposited on its account with the Minister, naming in detail the several securities, their par value, and value at which they are received as deposit;

Report before issue of licenses.

(*b*) In each case, before the issue of any new license, or the renewal of any license, make a report to the Minister that the requirements of the law have been complied with, and that from the statement of the affairs of the company it is in a condition to meet its liabilities;

Record of licenses.

(*c*) Keep a record of the licenses as they are issued;

Inspection of affairs.

(*d*) Visit the head office of each company in Canada, at least once in every year, and examine carefully the statements of the condition and affairs of each company, as required under this Act, and report thereon to the Minister as to all matters requiring his attention and decision;

Report to Minister of Finance, for Parliament.

(*e*) Prepare for the Minister, from the said statements, an annual report, showing the full particulars of each company's business, together with an analysis of each branch of insurance, with each company's name; giving items, classified from the statements made by each company;

Provision if Superintendent considers further enquiry necessary as to any company.

5. If the Superintendent, after a careful examination into the condition and affairs and business of any company licensed to transact business in Canada, from the annual or other statements furnished by such company to the Minister or for any other cause, deems it necessary and expedient to make a further examination into the affairs of such company and so reports to the Minister, the Minister may, in his discretion, instruct the Superintendent to visit the office of such company, to thoroughly inspect and examine into all its affairs, and make all such further enquiries as are necessary to ascertain its condition and ability to meet its engagements, and whether it has complied with all the provisions of this Act applicable to its transactions:

Books to be open to inspection.

6. The officers or agents of such company shall cause their books to be opened for the inspection of the Superintendent, and shall otherwise facilitate such examination so far as it is in their power; and for that purpose the Superintendent may examine under oath the officers or agents of such company relative to its business:

Record of inspection and report.

7. A report of all companies so visited by the Superintendent shall be entered in a book kept for that purpose, with notes and memoranda showing the condition of each company after such investigation; and a special report shall be communicated in writing to the Minister, stating the Superintendent's opinion as to its standing and financial position, and all other matters desirable to be made known to the Minister:

8. If it appears to the Superintendent that the assets of any company are insufficient to justify its continuance of business under the requirements of sections seven, eight, nine and ten, or that it is unsafe for the public to effect insurance with it, he shall make a special report on the affairs of such company to the Minister; and if the Minister, after full consideration of the report, and after a reasonable time has been given to the company to be heard by him, and upon such further enquiry and investigation as he sees proper to make, reports to the Governor in Council that he agrees with the Superintendent in the opinion so expressed in his report, the Governor in Council may, if he also concurs in such opinion, suspend or cancel the license of such company; and such company shall, during such suspension or cancellation, be held to be unlicensed, and unauthorized to do further business:

Special report if the company appear unsafe.

Proceedings thereon.

Suspension or cancellation of license.

9. Every person who, after notification of the suspension or cancelling of such license in the *Canada Gazette*, delivers any policy of insurance, collects any premium or transacts any business of insurance, on behalf of such company, shall be liable to the penalties provided for in the twenty-second section of this Act:

Penalty for carrying on business in such case.

10. Once in every five years, or oftener, at the discretion of the Minister, the Superintendent shall himself value, or procure to be valued under his supervision, the Canadian policies of life insurance of all companies licensed under this Act to transact the business of life insurance in Canada; and such valuation shall be based on the mortality table of the Institute of Actuaries of Great Britain, and on a rate of interest at four and one half per centum per annum, except in the case of bonus additions or profits accrued or declared before the twenty-eighth day of April, one thousand eight hundred and seventy-seven, and then valued on the basis of a rate of interest other than that above mentioned, which, in any such valuation, shall continue to be valued on such other basis:

Valuation of Canadian policies every five years.

Basis of valuation.

11. The Minister may, from time to time, instruct the Superintendent to visit the head office of any company licensed under this Act and incorporated or legally formed elsewhere than in Canada, and to examine into the general condition and affairs of such company; and if such company declines to permit such examination, or refuses to give any information necessary for such purpose, in its possession or control, its license shall be withdrawn:

Examination of affairs of a company out of Canada.

12. Every company now licensed, and every company hereafter licensed under this Act, and every company transacting life insurance business under the thirty-second section of this Act, shall annually contribute a sum in proportion to the gross premiums received by it

Payments by companies towards expenses of office of Superintendent.

in Canada during the previous year, towards defraying the expenses of the office of the Superintendent: which sum shall be paid upon the demand of the Superintendent:

Contribution by fire and marine companies limited

13. The sum to be contributed annually by companies carrying on the business of fire or inland marine insurance in respect exclusively of such business carried on by them shall not exceed in all eight thousand dollars:

Superintendent not to be interested in any company.

14. The Superintendent, or any officer or clerk under him, shall not, directly or indirectly, be interested as a shareholder in any insurance company doing business in Canada, or licensed under this Act:

Annual Report for Parliament.

15. The Minister shall lay the Superintendent's annual report before Parliament within thirty days after the commencement of each session thereof.

Enquiries by Superintendent.

25A. For the purpose of carrying out the provisions of this Act, the Superintendent of Insurance is hereby authorized and empowered to address any enquiries to any insurance company licensed under this Act, or to the president, manager, actuary or secretary thereof in relation to its assets, investments, liabilities, doings or conditions, or any other matter connected with its business or transactions, and it shall be the duty of any company so addressed to promptly reply in writing to any such enquiries.

PROVISIONS RELATING TO LIFE INSURANCE.

Provisions applicable to life companies

26. The provisions of sections twenty-seven to forty-three inclusive apply only to life insurance companies and to other insurance companies carrying on life and other insurance, only in so far as relates to the life insurance business of such companies.

CONDITIONS ON POLICIES.

Conditions not set out in full on the policy, to be void.

27. No condition, stipulation or proviso modifying or impairing the effect of any policy or certificate of life insurance issued after the first day of January, one thousand eight hundred and eighty-six, by any company doing business within Canada under the authority of the Parliament of Canada shall be good or valid unless such condition, stipulation or proviso is set out in full on the face or back of the policy.

As to statements in application for policy.

28. No policy or certificate shall contain or have endorsed upon it any condition providing that such policy or certificate shall be avoided by reason of any statement contained in the application therefor being untrue, unless such condition is limited to cases in which such statement is material to the contract.

Contract of life insurance not avoided by misstatement of age,

28A. Where in any contract of life insurance entered into with any company licensed to carry on business in Canada under the provisions of this Act, the age of

the person whose life is insured is given erroneously in any statement or warranty made for the purposes of the contract, such contract shall not be avoided by reason only of the age being other than as stated or warranted, if it appears that such statement or warranty was made in good faith and without any intention to deceive; but the person entitled to recover on such contract shall not be entitled to recover more than an amount which bears the same ratio to the sum that such person would otherwise be entitled to recover as the premium proper to the stated age of such person bears to the premium proper to the actual age of such person, the stated age and the actual age being both taken as at the date of the contract; but in no case shall the amount receivable exceed the amount stated or indicated in the contract.

if statement is made in good faith.

Effect of such statement.

2. For the purposes of this sub-section the word "premium" means the net annual premium calculated on the basis prescribed by this Act.

Interpretation. "Premium."

(The amending Act provides that this section shall apply to contracts of life insurance existing at the time of the passing of the Act, 23 July 1894, or thereafter entered into.)

Forfeiture and Renewal of Licenses.

29. Whenever satisfactory proof has been furnished to the Minister of any undisputed claim upon a company, arising on any policy of life insurance in Canada, remaining unpaid for the space of sixty days after becoming due, or of a disputed claim remaining unpaid after final judgment in a regular course of law and tender of a legal valid discharge made to the agent of such company, the Minister may withdraw the license of such company.

Withdrawal of license for non-payment of claims.

30. Such license may be renewed if within thirty days after such withdrawal such undisputed claims or final judgments upon or against the company are paid and satisfied.

Renewal if claim is satisfied.

31. When the license of a company carrying on the business of life insurance has been withdrawn by the Minister under any of the foregoing sections of this Act, such license may be renewed if, within thirty days after such withdrawal, such company complies with the requirements of this Act to the satisfaction of the Minister.

Renewal of license.

Companies Ceasing to do Business and Release of Deposits.

32. In the case of any company which, previously to the twenty-eighth day of April, in the year one thousand eight hundred and seventy-seven, was licensed to transact the business of life insurance in Canada, and which ceased to transact such business before the thirty-first

Provisions for case of certain companies which have ceased business.

Notice to Minister.

day of March, one thousand eight hundred and seventy-eight, having before that date given written notice to that effect to the Minister, the premiums due or to become due on policies actually issued before the last-mentioned date, may continue to be collected, and the claims arising thereon may be paid, and all business appertaining thereto may be transacted, and all proceedings appertaining thereto, either at law or in equity, may be continued or commenced and prosecuted; and the deposit at present in the hands of the Minister shall be dealt with under the law as it existed previously to the first-mentioned date, as if this Act had not been passed.

How deposit shall be dealt with in such case.

Power and proceedings of company ceasing business.

33. When any company licensed under this Act desires to discontinue business and to have its assets in Canada released, and gives written notice to that effect to the Minister, it may, with the consent of the policy-holders, procure the transfer of its outstanding policies in Canada to some company or companies licensed under this Act in Canada, or may obtain the surrender of the policies, as far as is practicable:

Use of assets in such case.

2. The trustees may employ any portion of the assets vested in them for the purpose of effecting such transfer or surrender:

List of policy-holders to be filed.

3. The company shall file with the Minister a list of all Canadian policyholders whose policies have been so transferred or have been surrendered, and also a list of those which have not been transferred or surrendered:

Notice to be published.

4. The company shall, at the same time, publish in the *Canada Gazette* a notice that will apply to the Minister for the release of its assets and securities on a certain day not less than three months after the date of the notice, and calling upon its Canadian policyholders opposing such release to file their opposition with the Minister on or before the day so named:

Action thereafter of Minister as to disposal of assets or securities.

5. After that day, upon the application for release being made, if the Minister, with the concurrence of the Treasury Board, is satisfied that such transfer or surrender has been effected, he may direct that a portion of the assets held by the trustees, or securities held by the Minister, shall be retained, sufficient in amount to cover the full equitable net surrender value of such policies (including bonus additions and accrued profits), as have not been transferred or surrendered, or in respect to which opposition has been filed, and may order the remaining assets or securities aforesaid to be released and transferred or paid over to the company:

Tenders to policyholders.

6. The portion retained shall be tendered in the manner hereinafter described to the aforesaid policy-holders *pro rata*, according to the aforesaid values of their respective policies; and on the acceptance of the amount so tendered, such policies shall thereby be

deemed to be cancelled: but if such tender is refused by any policyholder, the amount so tendered may be paid over to the company, and the policy shall continue in force; and such policyholder shall not be barred from any recourse he has, either in law or in equity, against the company to compel the fulfilment of its contract under such policy: If policyholders refuse the tenders.

7. The surrender-values above mentioned shall be determined by the Superintendent on the basis stated in the twenty-fifth section of this Act, and he shall collect from the company the expenses of such valuation at the rate of three cents for each policy or bonus addition, and shall pay the same to the Minister before the latter shall hand over the securities: Surrender-values, how determined.

8. Nothing herein contained shall prevent any policyholder from making special arrangements with the company whereby his policy may be continued in force; and, on proof being given of such arrangement, such policy may be omitted or removed from the abovementioned lists of policies, and this Act shall thereafter not apply in respect of such policy. Special arrangements may be made.

34. The tender referred to in the next preceding section shall be made in the following manner: How the tender shall be made.

(*a*) A list and notice in the Form D in the schedule to this Act, or to the like effect, shall be published in the *Canada Gazette* for at least thirty days previously to the day named in such notice; List and notice to be published in *Canada Gazette*.

(*b*) The company shall also cause the said list and notice to be published in such newspapers in Canada and for such length of time as the Minister determines; And in other papers.

(*c*) A notice in the Form E in the schedule to this Act, or to the like effect, shall be sent by mail (postpaid or franked) from the office of the Superintendent to each of the policyholders named in the said list, whose address is known to him, and such notice shall be deposited in some post office in Canada at least thirty days previously to the day named therein, which shall be the same day as that named in the list and notice above mentioned; Notice to be sent to each policyholder.

(*d*) Any policyholder who does not signify in writing to the Superintendent his acceptance of the amount so tendered, on or before the day named in the said notice, shall be deemed to have refused the same; but the Minister may, at any time prior to the payment over to the company of the amount so refused, allow any policyholder to signify his acceptance of such amount,—which acceptance, so allowed, shall have the same effect as if made on or before the day named in the said notice. Policyholders not signifying acceptance deemed to have refused.

2. In this and the next preceding section the word "policyholder" means the person to whom the policy is issued and with whom the contract for assurance is made, and includes the assignee of such person. "Policyholder" defined.

How reserve for covering liabilities to Canadian policyholders shall be calculated.

35. In computing or estimating the reserve necessary to be held in order to cover its liability to policyholders in Canada, each company may employ any of the standard tables of mortality as used by it in the construction of its tables, but there shall be set apart and credited to such reserve in each year out of the interest earned in the year, a sum equal to four and one-half per-cent per annum on the amount of the reserve as at the end of the preceding year, together with such further additions from premiums received during the year, if any, as shall be necessary to bring the reserve up to the standard provided by sub-section ten of section twenty-five of this Act: Provided, that in no case shall a company be required to maintain a reserve in excess of that provided for by the said sub-section ten of section twenty-five of this Act; but if it appears to the Superintendent that the reserve as computed by the company falls below that above provided for, he shall report the same to the Minister, who may thereupon direct the Superintendent to compute, or to procure to be computed under his supervision, the reserve on the basis therein mentioned, and the amounts so computed, if it differs materially from the return made by the company, may be substituted in the annual statement of assets and liabilities; and in such case the company shall furnish to the Superintendent, on application, the full particulars of each of its policies necessary for such computation, and shall pay to the Superintendent an amount at the rate of three cents for each policy or bonus addition so computed, which amount he shall pay over to the Minister.

Minister may order recomputation.

Cost.

Company may require computation by Superintendent.

2. Any company instead of itself computing or estimating the reserve above mentioned, may require the same to be computed by the Superintendent on the basis stated in the twenty-fifth section of this Act, on payment of a like amount as is mentioned in the next preceding sub-section:

Proviso: as to bonus additions or profits on policies.

3. Provided always, that in the case of any bonus additions or other profits on the policies of any company, accrued or declared before the twenty-eighth day of April, one thousand eight hundred and seventy-seven, and which have been heretofore valued on the basis of a rate of interest other than that above mentioned, such company may compute or have the same computed on such other basis; and provided also, that in the case of any company which has heretofore based its computation or estimate of its reserve necessary to cover its liability to policyholders in Canada (other than the reserve to cover the bonus additions or other profits in the last proviso mentioned) on a rate of interest of five per-cent per annum, the basis of computation or estimates mentioned in the twenty-fifth section and in

And as to companies having heretofore computed the reserve on 5 per-cent interest.

this section, shall not apply until the twenty-eighth day of April, one thousand eight hundred and eighty-seven, but such company may, until such date, compute such reserve, or have the same computed, at a rate of interest not exceeding five per-cent per annum.

Mutual or Assessment Life Insurance Companies.

36. No company shall carry on within Canada any business of life insurance by promising to pay on the death of a member of such company, a sum of money solely from the proceeds of assessments or dues collected or to be collected from the members thereof for that purpose without being licensed or registered under this Act, except that, in the case of any contract entered into, or any certificate of membership or policy of insurance issued before the twentieth day of July, one thousand eight hundred and eighty-five, by any company carrying on such business, assessments may be made and collected, and claims paid, and all business connected therewith transacted without any penalty being incurred.

Certain forms of insurance forbidden.

Proviso: as to contracts prior to this Act.

37. Any company incorporated or legally formed within Canada which transacts business of the nature described in the next preceding section may, at the discretion of the Minister, on report of the Superintendent approved by the Treasury Board, be exempted from the operation of the foregoing provisions of this Act, except those of sections twenty-five, twenty-seven, twenty-eight, twenty-nine, thirty and thirty-one, and be permitted to transact the business of life insurance on the conditions specified in the five sections next following.

Certain companies may be conditionally exempted from the operation of the Act.

38. Companies to be so exempted shall register their titles or corporate names in the office of the Superintendent; they shall also make attested returns of their condition and affairs at such times and in such form, and attested in such manner, as are prescribed by the Minister, and the Superintendent shall include such returns in his annual report: and any failure to make such returns, when called for by the Superintendent, shall subject such company, and any officer thereof, to the penalties mentioned in the twenty-first section of this Act:

Conditions of such exemption.

Penalty for non-compliance therewith.

2. The registration of any such company shall cease to be valid on the thirty-first day of March in each year, but shall be renewable from year to year, in the discretion of the Minister.

Yearly renewal of registration.

39. The provisions of this section shall apply to corporations or associations incorporated or legally formed elsewhere than in Canada for the purpose of carrying on the business of life insurance upon the co-operative or assessment plan:

Application of this section.

License may be issued on deposit of $50,000.

2. Any such corporation or association may be licensed by the Minister, under the provisions of this Act, to transact business in Canada upon depositing with him fifty thousand dollars, and thereafter shall have the right to transact business so long as it continues to pay its losses to the full limit named in its certificates or policies, and has complied with all the requirements of this Act and of the Superintendent of Insurance:

Duration of license.

Further deposits may be required.

3. In addition to such deposit of fifty thousand dollars, the Minister, upon the report of the Superintendent, approved by the Treasury Board, may, from time to time, require such other and further deposit as is recommended in such report and so approved, to be made by such companies or deposited with trustees to be named by the Minister, upon such trusts as are determined by the Governor in Council:

Death claims to be a first charge.

4. Death claims shall be a first charge on all moneys realized from assessments, and no deductions shall be made from any such death claims on any account whatsoever:

Application of moneys from assessments.

5. No portion of any moneys received from assessments for death claims shall be used for any expense whatever; and every notice of any assessments shall truly specify the cause and purpose thereof:

Notice to be printed on policy, &c.

6. Every application, policy and certificate, issued or used by any such company in Canada, shall have printed thereon in a conspicuous place, in ink of a colour different from that of the ink used in the instrument, and in good-sized type, the following words:—

Form.

"This association is not required by law to maintain the reserve which is required of ordinary life insurance companies":

Promise to pay out of certain funds to be contained in policy, &c.

7. Every certificate and policy shall contain a promise to pay the whole amount therein mentioned, out of the death fund of the association and out of any moneys realized from assessments to be made for that purpose, and every such association shall be bound forthwith and from time to time, to make assessments to an amount adequate with its other available funds, to pay all obligations created under any such certificate or policy without deduction or abatement:

In every policy issued in Canada.

8. The condition embodied in the next preceding subsection shall be inserted in every policy or certificate issued or delivered by any such company to any person insured in Canada:

Clause required in policies in favour of residents in Canada.

9. In every policy issued by a company licensed in accordance with this section of this Act in favour of a resident of Canada, a clause shall be either embodied therein or endorsed thereon, to effect that an action to enforce the obligation of such policy may be validly taken into any court of competent jurisdiction in the province wherein the policyholder resides or last resided before his

decease, and such policy shall not contain any provision inconsistent with such clause.

10. No company which is authorized to assure or assures to any of its members a certain annuity, either immediate or deferred, whether for life or for a term of years, or any endowment whatever, shall be eligible for license as an assessment company under the foregoing provisions of this Act. Assurance of annuities to debar from registration as assessment company.

11. No company shall be eligible for license as an assessment company: Conditions of such registration.

(*a*) If a new company, until it has received at least five hundred applications for membership calling for an amount of insurance not less than five hundred thousand dollars, the procuring of which applications shall not be deemed a violation of the provisions of section twenty-two of this Act: or In case of new company.

(*b*) If a company already engaged in business, unless it has at least five hundred members or policyholders holding policies for at least the sum of five hundred thousand dollars. In case of company in business.

40. The provisions contained in sub-sections four, five, six, seven, eight, ten and eleven of the next preceding section shall also apply to any company (not being such a company, society or association, as is referred to in section forty-three of this Act) incorporated in Canada and carrying on the business of life insurance upon the co-operative or assessment plan. Certain provisions to apply to companies not referred to in section 43.

41. The words "assessment system" shall be printed in large type at the head of every policy and every application for the same, and also in every circular and advertisement issued or used in Canada in connection with the business of a company to which any of the provisions of the five sections next preceding apply. Form of words to be printed on certain policies.

42. Every director, manager, agent or other officer of any such company as is hereinbefore lastly mentioned, which carries on business without being licensed or registered; and Penalty for doing business or using forms of policies, &c., in contravention of this Act.

(*b*) Every person who transacts any business of insurance on behalf of any such company which so carries on business, without being registered or licensed: and

(*c*) Every such company which neglects to print the words "assessment system" as provided by the next preceding section; and

(*d*) Every director, manager, agent or other officer of such company and every other person who transacts business on behalf of any such company and who circulates or uses any application, policy, certificate, circular or advertisement on which the words "assessment system" are not printed as hereinbefore provided, shall be liable to the penalties mentioned in the twenty-second section of this Act.

Certain societies exempted from this Act.

43. Nothing contained in this Act shall apply to any society or association of persons for fraternal, benevolent, industrial or religious purposes, among which purposes is the insurance of the lives of the members thereof exclusively; or to any association for the purpose of life insurance formed in connection with such society or organization and exclusively from its members, and which insures the lives of such members exclusively:

But such societies may avail themselves of this Act.

2. Any society or association which is declared by this section to be exempt from the application of this Act, may, nevertheless, apply to the Minister to be allowed to avail itself of the provisions of the seven sections next preceding, and upon such application being assented to, such society or association shall cease to be so exempt by virtue of this section.

Provisions Relating to Fire and Inland Marine Insurance.

What provisions shall be applicable to fire and inland marine companies.

44. The provisions of sections forty-five to forty-eight inclusive, apply only to fire and inland marine insurance companies and to other insurance companies carrying on fire and other insurance or inland marine and other insurance, in so far as relates to the fire or inland marine insurance business of such companies.

Forfeiture and Renewal of Licenses.

Licenses forfeited by failure to make deposit or non-payment of claims.

45 Whenever any company fails to make the deposits under this Act at the time required, or whenever written notice has been served on the Minister of any undisputed claim arising from loss insured against in Canada remaining unpaid for the space of sixty days after it becomes due, or of a disputed claim remaining unpaid after final judgment in a regular course of law and tender of a legal valid discharge, the license of such company may be withdrawn by the Minister.

Renewal on certain conditions.

46. Such license may be renewed, and the company may again transact business, if, within sixty days after notice to the Minister of the failure of the company to pay any undisputed claim, or the amount of any final judgment as provided in the next preceding section, undisputed claims or final judgments upon or against the company in Canada are paid and satisfied.

Companies Ceasing to do Business and Release of Deposits.

Duty of companies ceasing business.

47. When any company has ceased to transact business in Canada, and has given written notice to that effect to the Minister, it shall insure, on behalf of its Canadian policyholders, all its outstanding risks, in some company or companies licensed in Canada, or obtain the

surrender of the policies; and its securities shall not be delivered to the company until the same is done to the satisfaction of the Minister:

2. Upon making application for its securities the company shall file with the Minister a list of all Canadian policyholders who have not been so re-insured, or who have not surrendered their policies; and it shall at the same time publish in the *Canada Gazette* a notice that it has applied to the Minister for the release of its securities on a certain day, not less than three months after the date of the notice and calling upon its Canadian policyholders opposing such release to file their opposition with the Minister on or before the day so named; and after that day, if the Minister, with the concurrence of the Treasury Board, is satisfied that the company has ample assets to meet its liabilities to Canadian policyholders, he may order that all the securities be released to it or that a sufficient amount of them be retained to cover the value of all risks outstanding or respecting which opposition has been filed, and that the remainder be released; and thereafter, from time to time, as such risks lapse, or proof is adduced that they have been satisfied, further amounts may be released on the authority aforesaid.

Conditions on which deposits may be released.

Retention of amount to cover outstanding risks.

3. When a company has ceased to transact business in Canada after the notice hereby required, and its license has in consequence been withdrawn, such company may, nevertheless, pay the losses arising upon policies not re-insured or surrendered, as if such license had not been withdrawn.

Payment of losses after license has been withdrawn.

Fire Policies.

48. No fire policy shall be issued for or extend over a longer period than three years.

Duration of fire policies.

Insurance Other than Life, Fire or Inland Marine.

49. No company or person shall issue any policy other than a life, fire or inland marine insurance policy, or receive any premium in respect thereof, or carry on any business of insurance other than life, fire or inland marine insurance, without first obtaining a license from the Minister to carry on such business in Canada; the Treasury Board shall determine in each case what deposit shall be required to be made with the Minister, and the sections of this Act which shall apply to such company or person.

License requisite for transaction of business other than life, fire or inland marine insurance.

Deposit, and application of Act.

2. The Treasury Board, upon the report of the Superintendent, may revoke any such license if sufficient cause therefor be shown by such report.

Revocation of license.

Annual statements by licensed persons.

3. Any person receiving such license shall make annual statements under oath of such business at the same time and in the same form and manner as a company transacting the same business would under the provisions of this Act be required to make the same.

Powers of Superintendent.

Contribution towards expenses.

4. The Superintendent shall have the same powers with regard to a person receiving a license as are conferred on him by this Act with regard to insurance companies, and such person shall contribute towards the expenses of the office of the Superintendent a sum in proportion to the gross premiums received in Canada during the previous year.

Penalties for carrying on business mentioned in this section, without license.

5. Every company or person carrying on any such business without obtaining such license, or after such license is revoked, or neglecting or refusing to make the statements required, and every person who delivers any policy or insurance or collects any premium on behalf of such company or person shall respectively incur the penalties mentioned in the twenty-first and twenty-second sections of this Act.

Exception from section.

6. This section shall not apply to companies carrying on in Canada ocean marine insurance business exclusively.

[Then follow Schedules A to E, which are identical with those reprinted in *J.I.A.*, xxvii, 478-484.—Ed.]

58-59 VICTORIA.—Chap. 20.

An Act further to amend the Insurance Act.

[*Assented to* 22 *July* 1895.]

Her Majesty, by and with the advice and consent of the Senate and House of Commons of Canada, enacts as follows:

R.S.C., c. 124, s. 20, amended.

1. Sub-sections one and seven of the section substituted by section eight of chapter twenty of the Statutes of 1894, for section twenty of *The Insurance Act*, chapter one hundred and twenty-four of the Revised Statutes, are hereby repealed and the following substituted therefor:

Annual statement by companies incorporated elsewhere than in Canada.

"20. Every company incorporated or legally formed elsewhere than in Canada, and at present licensed or hereafter licensed under this Act, and every company which is subject to the provisions of this Act, shall make annual statements of its condition and affairs, at the

balancing day of the company in each year, and the form and manner of making such statements shall, as to the Canada business of such company, be the same, so far as applicable, as is required of Canadian companies, and as to its general business, shall be in such form as such company is required by law to furnish to the government of the country in which its head office is situate; and where such company is not required by law to furnish a statement to the government of the country in which its head office is situate, then such statement, as to its general business, shall be in such form as the company usually submits to its members or shareholders, and, in the event of no such statement being submitted to such members or shareholders, shall show in concise form the assets and liabilities of the company at such balancing day and the income and expenditure of the company for the year ending on such balancing day. The blank forms of the statements of the Canada business shall be supplied by the Superintendent." Forms for Canada business.

"7. The statements mentioned in the next preceding section, and the statements of Canada business provided for in the first sub-section of this section shall be deposited in the office of the Superintendent on the first day of January next following the date thereof, or within two months thereafter; and every statement of general business provided for in the said first sub-section of this section shall be deposited in the office of the Superintendent within fifteen days after it is required by law to be made to the government of the country in which the head office of the company whose statement it is, is situate, or within fifteen days after the submission of the same at the annual meeting of the shareholders or members of the company, whichever date first occurs: Provided, however, that no such statement of general business need be so deposited earlier than the first day of May, nor shall it be so deposited later than the thirtieth day of June next following the date thereof. The date of a statement in this sub-section referred to is the date at which the condition and affairs of the company are shown." Date for deposit of statement. Proviso. Date of statement.

2. The words "annual statement" in the section substituted by section eight of chapter twenty of the Statutes of 1894 for section twenty-one of *The Insurance Act* shall, in the case of companies incorporated or legally formed elsewhere than in Canada, be deemed to include both the statement of the Canada business and the statement of the general business provided for in the sub-section hereby substituted for sub-section one of section twenty of *The Insurance Act,* as amended by chapter twenty of the Statutes of 1894. Interpretation: "annual statement."

Renewals of licenses for 1895 confirmed.

Penalties remitted.

3. The renewals of licenses under *The Insurance Act* for the year now current are hereby confirmed, and any penalties incurred with respect to the statements of general business required to be filed in pursuance of the said Act are hereby remitted.

Section 39, ss. 2, to apply to assessment companies.

Proviso.

4. Notwithstanding anything in *The Insurance Act* contained, sub-section two of section thirty-nine thereof shall apply to companies incorporated or legally formed in Canada for the purpose of carrying on the business of life insurance on the assessment plan: Provided, that this section shall not interfere with the renewal of certificates of registration heretofore granted.

R.S.C., c. 124, s. 4, amended.

5. Section four of *The Insurance Act* is hereby amended by adding the following sub-section thereto:—

Name of company to be approved by Minister.

"2. Before issuing a license to a company legally formed elsewhere than in Canada, the minister must be satisfied that the corporate name of the company is not that of any other known company incorporated or unincorporated, or any name liable to be confounded therewith or otherwise on public grounds objectionable."

THE INSTITUTE OF ACTUARIES.

Additions to the Library.

The following works have been added to the Library since the publication of the new Catalogue:

	By whom presented (when not purchased).
Actuarial Society of Edinburgh.	
Laws of.	*Actuarial Society.*
Actuarial Society of America.	
Papers and Transactions, Vol. 3.	*Actuarial Society.*
Attorneys and Solicitors.	
Charter of Incorporation, Bye Laws, Regulations, Grant of Arms, and the Statutes relating to. London, 1845.	
Australian Mutual Provident Society.	
Case for Opinion and Opinions thereon, by Messrs. Hendriks, A. H. Bailey, and R. P. Hardy, and report of General Manager. 1893.	*A. M. P. Society.*
Baily (Francis).	
The Doctrine of Interest and Annuities analytically investigated and explained. 1808.	
The Same. Edited by Filopowski. Edinburgh, 1866.	*Anonymous.*
The Doctrine of Life Annuities and Assurances. Two Copies. Liverpool, 1864.	*Anonymous.*

	By whom presented (when not purchased).
Baines (A.) M.I.J.	
The Popular Hand-Book of Finance.	
Barlow (Peter).	
Tables of Squares, Cubes, &c. London, 1856.	*Anonymous.*
A new Mathematical and Philosophical Dictionary. 1814.	*T. G. C. Browne.*
Bland (Rev. Miles).	
Algebraical Problems, with solutions. Seventh edition. Cambridge, 1837.	*Anonymous.*
Becher (Rev. J. T.).	
The Constitution of Friendly Societies Exemplified by the Rules and Tables of the Southwell Friendly Institution; with Observations on the rise and progress of Friendly Societies. Fifth edition. 1829.	
Belgium.	
Assurances sur la vie. Assurances Contre Les Accidents. Rapports présentés par, M. H. F. G. Adan, à l'Exposition Universelle de Paris. 1889.	*The Author.*
Assurances Contre Les Accidents. De la Responsabilité Civile des Patrons de l'article 1382 du code civil et de la Faute Lourde en Matiére D'Assurances. Paris, 1883.	*The Author.*
Coup d'oeil sur le Projet de Révision du Code de Commerce au point de vue des Assurances sur la vie. Par H. F. G. Adan. Bruxelles, 1870.	*The Author.*
Caissé centrale de Prévoyance des Secrétaires communaux. 1893.	*H. F. G. Adan.*
La Belgique Judiciaire. Five numbers.	*H. F. G. Adan.*
Compte Rendu des Operations et de la situations de la Caissé Générale D'Epargne et de Retraite. 1893.	*M. Mahillon.*
Considerations Pratiques sur l'examen medical en matiére d'Assurances sur la vie, by Dr. A. Mahillon. Bruxelles, 1894.	*The Author.*
Text Book de L'Institut des Actuaires de Londres, Part II, Traduction de Mons. Amédée Bégault. Bruxelles, 1894.	*The Translator.*
Bourne.	
Handy Assurance Manual, 1894.	
Handy Assurance Directory, 1894.	
Bourne (J. P.).	
Life Interests or Annuities. Policy Values. Reversionary Values.	*The Author.*
Castaing (J.).	
Interest Tables, 1708.	
Cleghorn (James).	
Reports on the Widows' Funds of Writers to H. M. Signature.	
Reports on the Widows' Funds of the Faculty of Advocates.	
Cracroft's Trustees' Guide, 12th Ed., N. D,	*T. G. C. Browne.*

By whom presented (when not purchased).

Deuchar (D.).
Notes on Widows' Funds. Edinburgh, 1894. — *The Author.*

Digest of Cases and Decisions under the Employers' Liability Act, 1880.
London, 1893. — *T. J. W. Buckley.*

Felkin (R. W.) M.D.
On the Geographical Distribution of Tropical Diseases in Africa.

Germany.
Die Finanzlage der Gothaischen Staatsdiener-Wittwen-Societät am 31 Decr. 1890. Von Professor Johannes Karup. Dresden, 1893.
Deutsche Sterblichkeits-Tafeln. Berlin, 1893.

Giles (John).
Free Trade in Gold: Being a reply to the Cobden-Chevalier Treatise, "on the probable Decline in the Value of Gold," also an exposition of the French Schemes on the Currency now maturing. London, 1861. — *T. G. C. Browne.*

Griffith (Alexander).
Cycle Calendar; or, 128 years Cycle System of Measuring Time. London, 1892. — *The Author.*

Hearts of Oak Benefit Society.
Report upon the Results of a Valuation of the Society's Liabilities and Assets as at 31 December 1892. — *R. P. Hardy.*

Hewat (A.).
On an Investigation of the Marriage and Mortality Experience of the Widows' Funds of the Scottish Banks; with the relative tables and diagrams; and Notes on other Marriage and Mortality Tables. London, 1895. — *The Author.*

Hind (John).
A Digested Series of Examples in the application of the principles of the Differential Calculus. Cambridge, 1832. — *Anonymous.*

Holland.
Jaarboekje van der Vereeniging voor Levensverzekering. 1891, 1892, and 1893. — *E. W. Scott.*
Rechten op de Zoogenaamde Reserve eener Levensverzekering. Ondememing Door C. L. Scott. Amsterdam, 1894. — *E. W. Scott.*
Wettelijk Karakter van het Contract van Levensverzekering. I. van Schevichaven. Amsterdam. — *E. W. Scott.*

Insurance Blue Book and Guide for 1894-5. London, 1894. — *Champness & Co.*

Italy.
Bilancio Tecnico della Società di Mutuo Soccorso in Napoli detta Fondo del grano. Roma, 1894. — *The Author.*

Jones (C. N.).
A Life Table based upon the Experience of the New York Life Insurance Company within the American Tropics. C. N. Jones. N. Y., 1894. — *The Author.*

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Life Assurance.	
Popular view of. Anonymous. 1840.	
Life Offices Association.	
Reprint of Papers, 1889 to 1894.	*T. E. Young.*
London & North Western Railway Company.	
Superannuation Fund Association. Actuaries' Reports, 1888, 1893.	*Hy. Cockburn.*
Manchester, Insurance Institute of.	
Report, 1892-3.	*The Institute.*
Monilaws (W. M.).	
The Surplus Funds of Life Assurance Offices. 1893.	
Life Assurance. Companion to Surplus Funds. 1894.	*Hy. Cockburn.*
The Surplus Funds and Policy Conditions of Life Assurance Offices. 1887.	
New South Wales, Insurance Institute of.	
Proceedings. 1893.	*The Institute.*
Northampton Tables.	
Fol. MS. N.D.	
Official Year Book of Scientific and Learned Societies—	
1894, 1895.	
Parliamentary Papers.	
Acts of Parliament, sundry. Geo. III, William IV.	
Acts of Parliament, sundry. Victoria.	
Colonies.	
Australasia,	
The Seven Colonies of. Statistical Account. 2 vols. 1893, 1894.	*The Government Statistician.*
New South Wales.	
Statistician's Report on the eleventh Census of.	
New Zealand.	
Annual Report of the Government Insurance Department. 1893.	*N. Z. Government.*
Official Year Books, for 1893 and 1894.	*N. Z. Government.*
Reports on Statistics, 2 vols. 1889 and 1890.	*N. Z. Government.*
Statistics of the Colony of, 1893.	*N. Z. Government.*
Victoria.	
15th Annual Report of the Proceedings of the Government Statist in Connexion with Friendly Societies. 1892.	*The Government Statist.*
Statistical Register. 1892.	*The Government Statist.*
Friendly Societies.	
Abstract of the Quinquennial Returns of Sickness and Mortality experienced by Friendly Societies for the five years 1845–50.	
Acts 1875–6–9.	
Report from Select Committee on No. 2 Bill. 1893	
Income Tax.	
Second Report from the Select Committee, 1852.	
Reports of Royal Commission on Vaccination, 1 vol. 1889–90.	
Savings Banks Return. 1888.	
Superannuation of Teachers in public and elementary schools. Report 1894.	

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Periodicals.	
Institute of Bankers' Journal.	*The Institute.*
Insurance File, 2 vols., 1892, 1893.	*The Publishers.*
Insurance Record, Vols. 31, 32, 1893, 1894.	*The Editor.*
Insurance Register, 1894.	*Messrs. C. & E. Layton.*
Insurance Spectator of London, Vol. 16, 1893.	*The Editor.*
Insurance Year Book, 1889. New York.	
London Mathematical Society's Journal.	*The Society.*
Post Magazine, 1894.	*The Editor.*
Royal Statistical Society's Journal.	*The Society.*
Zeitschrift für Versicherungs-Recht und Wissenschaft. Bd. 1. Heft 1.	*The Editor.*
Premium Conversion Tables reprinted from the Insurance Spectator of London.	*Editor of Insurance Spectator.*
Ratcliffe (Henry).	
Independent Order of Odd-Fellows, Manchester Unity Friendly Society Supplementary Report, 1 July 1872.	
Russia.	
Bulletin Russe de Statistique Financière et de Legislation, 1ère Année, 9 Nov. 1894.	*Russian Minister of Finance.*
Shearwood (J. A.).	
Abridgment of the Law of Real Property, by J. A. Shearwood. 3rd. Ed. London, 1885.	
Shortrede (Robert).	
Logarithmic Sines and Tangents, with Arguments in Space and Time, 1849.	
The Same, New Edition, 1858.	
Snowball (J. C.).	
The Elements of Plane and Spherical Trigonometry. 5th Edition. Cambridge, 1840.	
Solicitors' Remuneration Order, 1889.	*T. G. C. Browne.*
Strachan (T. Y.).	
Report upon a proposed fund for relief in cases resulting from Accidents and in Sickness and Old Age among the employees in the metalliferous mines of Devon and Cornwall and District. Newcastle-on-Tyne, 1888.	*The Author.*
The Insurance of Securities. A Lecture delivered to the Chartered Accountants' Students' Society of London. London, 1893.	*The Author.*
Switzerland.	
Rapports du Bureau Féderal des Assurances sur les Enterprises Privées D'Assurance en Suisse. 1892 and 1893.	*Swiss Government.*
Symms (W.), Executor of the late Thomas Sinclair.	
The Scottish Widows' Fund Life Assurance Society. Belfast, 9 March 1868.	*Dr. T. B. Sprague.*
Turnbull (A. D. Lindsay).	
Interest and Annuity-Certain Logarithm Card, 1895.	*The Author.*

By whom presented (when not purchased).

United States.
New England Mutual Life Insurance Company of Boston, Mass. Female Life Mortuary Experience from 1863-1892. — *W. C. Wright.*
New York Insurance Report. Parts II, III and IV. Life, Casualty and Assessment, 1895. — *N. Y. Life Office.*
Hand Book of Life and Accident Insurance on the Assessment Plan. New York, 1894.
The Insurance Year Book, 1894-5. (Life and Miscellaneous.) New York, 1894.

Victoria, Insurance Institute of.
Reports, 1893, 1894. — *Institute.*

Watson (Reuben.)
The Fifth Valuation of the Lodges of the Manchester Unity Friendly Society, 1889-91-93.

Williams (Joshua).
Principles of the Law of Real Property. 8vo. 17th Edition, 1892.

Wolstenholme (E. F.) and Turner (R. O.).
The Conveyancing Acts, 1881 and 1882, with Notes, Forms, and Precedents. London, 1882. — *T. G. C. Browne.*

Yorkshire, Insurance Institute of.
Proceedings, 1893-4. — *Institute.*

The following duplicate copies of works already in the Library have also been added:

By whom presented (when not purchased).

Actuaries, Institute of.
Charter and Bye Laws. 1884.
Constitution and Laws. List of Members and Catalogue. 1849, 1850, 1851.
Mortality Experience of Life Assurance Companies, collected by the Institute. 1869. — *Anonymous.*
Text-Book, Part I. Interest and Annuities-Certain, by William Sutton. 1882. — *Anonymous.*

Chisholm (David).
Commutation Tables for Joint Annuities and Survivorship Assurances, based on the Carlisle Table of Mortality at various Rates of Interest. Vol. I. London, 1858. — *Anonymous.*

Davies (Griffith).
Treatise on Annuities, with Tables (posthumous), N. D. — *Anonymous.*

Gray (Peter).
Tables and Formulæ for the Computation of Life Contingencies.
Second issue with an addendum comprising a large extension of the principal Table. 8vo. London, 1870. — *Anonymous.*

Hardy (Peter).
New and General Notation for Life Contingencies. 1840. — *Anonymous.*

By whom presented (when not purchased).

Jones (Jenkin).

A Series of Tables of Annuities and Assurances, calculated from a new Rate of Mortality amongst Assured Lives: with examples. London, 1843. — *Anonymous.*

Milne (Joshua).

A Treatise on the Valuation of Annuities and Assurances on Lives and Survivorships, on the construction of Tables of Mortality, and on the probabilities and Expectation of Life. 2 vols. London, 1815. — *Anonymous.*

Orchard (William).

Single and Annual Premiums for every value of Annuity, adapted to any Mortality Table. Lon. 1856. — *Anonymous.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

The International Congress of Actuaries of 1895, *at Brussels.* *By* A. J. FINLAISON, C.B., *President.*

[Read before the Institute, 25 November 1895.]

THE First International Congress of Actuaries that has been held since the profession of Actuary has been generally recognised, was opened on 2 September 1895, at the Palais des Académies, at Brussels, under the patronage of the Belgian Government, by the Belgian Finance Minister, M. de Smet de Naeyer, who had previously acted as honorary president of the organizing committee. This organizing committee had skilfully performed the laborious duty of collecting the materials and settling the procedure which so largely contributed to the success which attended the meetings of the Congress.

The preliminary work required for the successful organization of an International Congress is no light task. It begins many months before the first meeting, and the occupation of the officials does not conclude at the culmination of affairs which takes place at the meeting of the Congress. Those who knew the energy and capability of the corresponding members of the Institute of Actuaries resident in Brussels, of M. Mahillon, of

M. Bégault, the translator of the second part of our text-book, of M. Adan, and of their friend, M. Le Jeune, were not unprepared to find the business of the Congress efficiently carried out. Your representatives were acquainted with the merits of these gentlemen, and they subsequently learnt the capabilities of those who had been associated with them by finding that all things essential to the success of the Congress had been foreseen.

The assembly, at the opening, included delegates who had been nominated by the Governments of France, Belgium, Holland, Russia, Sweden, Portugal, and Luxembourg. It also included delegates from the leading Institutes and Societies of Actuaries; from the Institute of French Actuaries of Paris came M. Marie, M. Martin-Dupray, and M. Quiquet. The Institute of Actuaries of London sent Messrs. Finlaison, King, Manly, Chisholm, and Ryan. The Faculty of Actuaries of Edinburgh were represented by Dr. Sprague and Mr. Meikle, and the Actuarial Society of America by Messrs. McClintock, Homans, Pierson, St. John, and Macaulay.

Many delegates were also present from Insurance Societies from all parts of Europe, from America, Africa, and even from far Japan, while among them in the crowded hall were members of various nationalities who had voluntarily given their support to the objects of the Congress.

First Day, 2 September 1895.

After the opening speech of the Finance Minister, the meeting proceeded to the nomination of M. Mahillon as President of the Congress, of M. Bégault as Secretary, and of M. Duboisdenghien as Treasurer.

The President of the Institute of Actuaries was appointed a Vice-President for England, the President of the Faculty of Actuaries as Vice-President for Scotland, the President of the Actuarial Society of America as Vice-President for America, and leading representatives from other countries as Vice-Presidents for their nationalities. Mr. George King was appointed Secretary for the English-speaking nationalities, M. Grosse for Germany, and M. Paraira for Holland.

The Congress then proceeded to the consideration of the first paper on the programme, the author being M. Lepreux, of the Belgian Governmental Savings Bank and Office for Life Annuities payable during old age. The subject was the important

one of "The measures which can be taken to provide against the consequences of variations in the rate of interest by institutions that contract engagements continuing for a long term of years."

The paper commenced with an able review of the progressive reduction which had taken place throughout Europe in the rate of interest during almost half a century. M. Lepreux was of the opinion of M. Rouvier, the French Minister, that Government debts were the criterion of the value of all other securities. French 3 per-cents stood at 70 at the end of the year 1869, now they were over 102, and German, Prussian, Dutch, Swiss and Belgian 3 per-cents all now stood at prices above 100. Fifteen years ago Belgian Municipal 3 per-cent loans were quoted at $81\frac{1}{2}$, now they were $98\frac{1}{2}$. The rate of interest on Mortgages in France used to be 5 per-cent and even more, now it was 4 per-cent and, in some cases, $3\frac{3}{4}$ per-cent. The rate of interest returned by good house property used to be 5 to $5\frac{1}{2}$ per-cent in France, now it was 4 per-cent at the most.

M. Lepreux was of opinion that the continuous reduction in the rate of interest would weigh more heavily upon the whole-life engagements of assurance societies than upon their endowment assurance risks. It would seriously increase the engagements in respect of deferred life annuities payable either by governments or funds formed by mutual association, and it would further involve the position of the widows' pension funds in Belgium, which was already no secret.

The principal railway Company in France, which had a fund for the provision of pensions to their servants, based upon the total length of service and the average pay of the final six years, began with demanding a contribution of 4 per-cent of the pay from the men, while to this the company added an amount equivalent to 3 per-cent of the men's pay. But the growth of the charge for pensions and the steady reduction of the rate of interest receivable by the fund, obliged the company to raise the subvention they gave successively to 4, 6 and 8 per-cent, until, in 1892, it was necessary to raise the total contribution to 14 per-cent of the pay. Even this has been found insufficient, and it is now estimated that 16 per-cent of the pay is the least that will be possible to sustain the fund. It has consequently been decided to raise the contribution of the servants to 6 per-cent of their pay and the subvention of the company to 10 per-cent of the pay.

The figures from the paper of M. Lepreux have been quoted at

some length in the hope that their publication may give occasion for reflection to those public bodies and private institutions in this country who appear to think that adequate Superannuation Funds can be formed on much less onerous terms than those given above. There is good reason to believe, however, that an under-estimate of the vitality of the officials and of the power of the prospect of a pension to retain the members of the fund in the service of the Company has had more to do with the apparently unexpected charge for pensions than the variation of the actual from the anticipated receipts from interest.

M. Lepreux proceeded to state that it was well known that the reduction of the rate of interest has been one of the causes which have determined the French Life Assurance Companies to raise their rates of premium.

The effect of the reduction of the rate of interest has been increased by the endeavour of States and other bodies to convert their debts from the original to a lower rate of interest. The power of private bodies to forcibly effect such a conversion before the stipulated time for repayment of the debt is now being argued before the French Law Courts.

The discussion which subsequently arose on the paper was principally directed to this latter, and, at the present moment, important point. It was, however, the means of leading the discussion towards the legal interpretation to be placed upon the terms in which the well-known bonds of the railways and municipalities of France have been drawn. Dr. Sprague endeavoured to lead the discussion back to the titular subject of the paper by concisely explaining to the meeting the principles upon which the premiums of British insurance companies were based, that is to say, on a rate of interest which is estimated to be obtainable at the least for a long series of years. This line of argument was also expanded at length in a printed note by Mr. Fackler, formerly President of the Actuarial Society of America, but the immediate importance of the possible legal powers of debtors who were already occupied in endeavouring to reduce the rate of interest payable by them, proved of greater attraction than the discussion of the general question, and the sitting concluded with the decision that the Congress should not express any definite opinion upon the subject of the paper.

Second Day, 3 September 1895.

The proceedings of Tuesday, 3 September 1895, were perhaps the most important of the whole meeting, and were especially interesting to members of the Institute of Actuaries. The first paper on the programme was one by M. Martin-Dupray upon the means for the exchange of publications between actuaries of various countries.

The President of the Institute of Actuaries, the President of the Faculty of Actuaries, and the President of the Actuarial Society of America, each explained to the Congress that it was already their practice to exchange their Journals with other bodies who sent them accounts of the proceedings of their Societies.

M. Quiquet, of Paris, proposed that a permanent committee of the Congress should be nominated who should annually publish in a single collection all the actuarial communications which might be sent to it. The proposition was favourably received by the Congress, and the following committee were subsequently nominated :

Germany ..	MM. Samwer.
	Gerkrath.
	Grosse.
	Bödiker.
Austria-Hungary	Raffmann.
	Scholtz.
Belgium ...	Mahillon.
	Adan.
	Bégault.
	Duboisdenghien.
France ...	Guieysse.
	Martin-Dupray.
	Marie.
	Cheysson.
Great Britain	Finlaison.
	King.
	Ryan.
	Sprague.
	Meikle.
Holland ...	MM. Wolterbuck.
	Scott.
	Paraira.
Italy	Perozzo.
	Coja.
Russia	de Savitch.
	Paul.
Switzerland	M. de Cérenville.
Sweden	Linstedt.
United States of America	MM. Mc Clintock.
	Pierson.
	Homans.
Canada	Macaulay.

It will be evident, however, to the members of the Institute of Actuaries that a society established for the purpose above-mentioned, would have to proceed with very great caution in view of the copyright laws of the various countries of Europe.

The President, M. Mahillon, proposed that the resources of the Society be dependent upon an annual contribution from each member and the generosity of assurance companies.

The proposal was adopted unanimously by the Congress.

The second paper of the day, by M. Maingie, of Brussels, upon the necessity of introducing the study of the principles of life assurance into the programme of instruction in schools and colleges, was the occasion of several eloquent orations by various speakers, who appeared to consider that classes of this nature would be valuable to students.

The Congress then passed on to the consideration of a masterly paper by M. Bégault upon the necessity of adopting a universal notation for use in actuarial publications.

It had become a matter of general knowledge that M. Bégault was a warm advocate of the adoption of the notation of the Institute of Actuaries as the one most suitable for universal use. This fact had apparently been the means of inducing some of the members of the Congress, who had not had the opportunity of fully studying the merits of the notation, or of comparing it with other systems which have from time to time been in vogue, to promulgate that they were in favour of deferring the adoption of any system of notation, until a dictionary, or synopsis of all the notations in use had been prepared and submitted to the Congress. It was generally understood that a proposal would be made at Brussels that such a dictionary should be prepared under the supervision of a committee to be appointed by the Congress, as an alternative to a resolution that would recommend the adoption of any particular system.

The knowledge that such a proposition was likely to be made stimulated M. Bégault to prepare beforehand a comparison of systems of notation for the use of the Congress, and notwithstanding the comparatively short time at his disposal and the large amount of other work he had undertaken in connection with the Congress, he accomplished this self-imposed task in the most successful manner.

M. Bégault stated in his paper that it would be neither the intention, nor would it be in the power, of the Congress to impose a notation upon the actuaries of different countries, but, if it was recognized that the notation which the Congress preferred was simple, rational, and complete, it was to be hoped that it would be employed in preference to all others. He then

proceeded to exhibit in a synoptical table the principal notations in use throughout the world, that is to say:

1. The notations employed by English and American actuaries, or in other words, by the Institute of Actuaries and the Actuarial Society of America.
2. The notations used by the French actuaries, *i.e.*, the notation of Dormoy and that of the Committee of four French Life Assurance Companies.
3. The notation of Karup, père, used in Germany.
4. The notation of C. Landré, employed in Holland.

The author of the paper noticed that, owing to the desire to adopt the initial letter of a word as the symbol of the idea represented by it, the notations numbers 1 and 2 had many agreements, while the same thing was to be observed in the notations 3 and 4.

This arose from the English actuarial verbiage being nearly entirely of Latin origin. The rule of adopting the initial letter of the word desired to be represented could not be altogether adhered to, in view of the large number of terms employed, consequently, in the case of the less used operations the letters adopted did not in any way recall the quantities which they represented. This would explain the impossibility of taking part of the symbols of one system of notation and part of those of another to form a fresh system, for the letters adopted in one system would very likely be found to be taken in a different meaning in another.

M. Bégault went on to demonstrate in the most skilful manner the superior descriptiveness of several of the symbols used in the notation of the Institute of Actuaries above the symbols used in other systems, and concluded his paper by affirming that the notation adopted by the Institute of Actuaries of London, the Faculty of Actuaries of Edinburgh, and the Actuarial Society of America, was rational, and did not harass the memory. It was complete while it permitted an extension to the most complicated combinations. He proposed that the Congress should pass a vote.

1. That the notation of the Institute of Actuaries should be employed by the actuaries of all countries.
2. That any modifications that might be found in future to be necessary in the above-mentioned notation should be discussed in future International Congresses.

The eloquent address of M. Bégault appeared to carry conviction to the minds of the members present, and the paper was followed by a comparatively short discussion.

Mr. McClintock supported the proposition of M. Bégault, which he said was already realized as far as America was concerned.

M. Léon-Marie, Secretary of the Institut des Actuaires Français, also agreed to the proposal, while he considered that some modifications of the English notation might be made.

Mr. George King supported the proposal, not from national prejudice, but because he really believed the English notation to be the most perfect, for the reason that actuarial science and the notation in relation to it had been longer discussed in England than in any other country.

An exchange of ideas then took place between the President of the Congress and M. Quiquet, who was in favour of leaving the matter in the position in which it then stood, but who finally agreed with all the members present in unanimously voting that:

> The English notation should be adopted by the Congress, who invited actuaries to use it until the following Congress, when any modifications which might be proposed to be made to it could be discussed.

The unanimous adoption of this resolution by the Congress was considered by the Members of the Institute of Actuaries who were present as a matter of congratulation and a recognition of the work of the Institute in the cause of actuarial science. The Delegates of the Institute were especially gratified that this vote was taken in the presence of Dr. Sprague, who, while he was at the Congress in the capacity of President of the Faculty of Actuaries of Edinburgh, had formerly taken so large a part, when he was resident in London, in the labours of the Committee of Actuaries which developed, between the year 1869 and the year 1872, the scheme of notation that was published in 1872 as an appendix to the tables deduced from the mortality experience of Life Assurance Companies collected by the Institute of Actuaries.

It will be in the minds of the members present this evening that the recognition which the Institute of Actuaries received at the Congress from the representatives of the actuarial profession throughout the world, was due in great measure to the author of the second part of the *Text-Book*, Mr. George King, to the

admirable translation of the work into the French language by M. Bégault, and to his friends, MM. Mahillon and Le Jeune, who rendered so much aid in his task.

Third Day, 4 September 1895.

The proceedings on the third day commenced with a paper by M. Duboisdenghein, Secretary of the Belgian Governmental Savings Bank and Life Annuity Department, upon the methods by which errors prevalent in relation to provident institutions could be exposed.

The paper began with a reference to the various errors of principle upon which many of the pension funds for annuities in old age or for the benefit of widows or orphans had been established in Belgium. The remedy, M. Duboisdenghein thought, was to be found in the formation of an Institute which would follow in the footsteps of the Institute of Actuaries of London, of the Faculty of Edinburgh, of the Institute of French Actuaries, and of those Universities in Germany, Austria, and Switzerland, who held examinations and afforded students the means of education in actuarial subjects.

M. Vellut, of Brussels, in speaking upon the paper, said he considered that the principles of life assurance should be taught in all schools, and to both sexes, from the elementary schools upwards.

M. Marie, of Paris, thought great responsibility attached to governmental and municipal departments, and railway companies, who appeared to think they could found and successfully manage the provident societies attached to their establishments without the assistance of competent actuaries. These funds now required the services of actuaries to re-organize them from top to bottom.

The second paper of the day was presented to the Congress by M. Adan, of Brussels, one of the corresponding members of the Institute of Actuaries, upon the necessity for special legislation recognizing the general principles of life assurance.

M. Adan, who is a doctor of law as well as an actuary, stated that in England and Scotland a life assurance contract was recognized by the law, and notably so in the Married Women's Property Acts of 1870 and 1882, of England, and the Married Women's Policies of Assurance Act of 1880, of Scotland. Life assurance policies were also recognized by the law in the United

States of America; and in Germany, where they were legalized as far back as February 1794. In France, however, life assurance contracts were struck out of the Code Civil by the Legislature of 1804, as if life assurance was to be considered to be of the nature of a gambling transaction.

M. Adan then referred to the difficulties which arose in countries where French law prevailed in the division of the money arising from a life assurance policy; that is to say, whether it should be freely disposable by the assured person, or whether it should fall into his heritable property to be divided among his relations according to the law of France. M. Adan concluded by expressing a hope that the Belgian Government would appoint a committee to draft a life assurance law.

The paper was then discussed with much interest by French, Belgian and Dutch Members of the Congress. Several English-speaking members did not at first see their way to join in a vote which would seem to imply in any way that they wished for a change in the laws of their respective countries on the subject of life assurance policies, with which they expressed their general satisfaction, but, on it being understood that the vote would refer solely to countries under the French law, an unanimous vote was given by the Congress in favour of the proposal contained in the paper.

The third paper before the Congress on 4 September was on the subject of Select Mortality Tables, by M. Marie of Paris, who, with many of the Continental members, were much interested in a speech by Mr. G. King in which he acquainted the Congress with the attention that had been given and the publications which had been made in Great Britain by Dr. Sprague and others upon the subject.

Fourth Day, 5 September 1895.

The business on the fourth day began with an introduction by M. Quiquet, of Paris, of a proposition made by the Institute of French Actuaries and developed in a paper by M. Cheysson, of that Institute, which was also read by M. Quiquet. The proposition affirmed the desirability of provident societies being regulated upon proper principles and conducting their affairs under the advice of competent actuaries.

The paper in great measure went over the ground occupied by

M. Duboisdenghien on the previous day, and the Congress passed on to the consideration of the second paper of the day, by Dr. Sprague, upon a question which his able treatment has brought into prominent notice, namely: the explanation of a method of valuation which takes account of the expense of obtaining new business.

In the course of the paper, which was listened to with very great interest, Dr. Sprague gave expression to several interesting conclusions at which he had arrived from the published accounts of British offices; for instance, that the average expenditure for management of prosperous and well managed offices may be taken as 80 per-cent of the new premiums and $7\frac{1}{2}$ per-cent of the renewal premiums, further that most offices are still able to realize an average rate of interest of 4 per-cent per annum. Upon the adoption of these deductions he showed that, if an office made 4 per-cent per annum interest and experienced a rate of mortality corresponding to his select mortality tables, published in volume xxii of the *Journal of the Institute of Actuaries*, the sums in hand on account of policies taken out, for instance, at the age of 40 years with less than four years' standing, would not, with their accumulations at 4 per-cent per annum compound interest, equal the reserve required by the orthodox H^M 3 per-cent net premium valuation. He considered that this result demonstrated that the net premium system of valuation is erroneous, and advocated that the gross premiums should be used, when they had been reduced to an extent that would make the reserve at the end of the first year of assurance correspond to the hypothetical fund that would be in hand according to the Select Tables.

In the discussion which followed, Mr. Homans said that he thought that the preferable way was to treat the first year of assurance as a temporary risk, and to consider the life risk to run from the date for the payment of the second premium.

The next paper was presented by Mr. H. R. Harding, of the Economic Life Assurance Society, on the legislation in reference to life assurance companies in various countries. The paper was read to the Congress by the Secretary, M. Bégault. An interesting discussion arose, in which the arbitrary character of the interference to which both native and foreign companies were subjected in Prussia was brought to the notice of the Congress by M. Adan, of Brussels, and Mr. McClintock, of New York.

Dr. Grossman, of Vienna, informed the meeting that the

Austrian legislation, with regard to foreign companies, would be absolutely prohibitive if it were put in force, but that, as a matter of practice, it only existed on paper.

FIFTH AND LAST DAY, 6 SEPTEMBER 1895.

The final day of the Congress was devoted in the first place to a paper by Mr. McLauchlan, of the Faculty of Actuaries, upon the arrangement of the census returns of different countries. The paper was read in abstract to the meeting by Dr. Sprague. It had reference principally to the arrangement of the abstracts of the census of the United Kingdom of 1891, and the variations of this arrangement from those previously adopted.

Mr. G. H. Ryan said it was a mistake to think actuarial science had to deal exclusively with life assurance, and it was a great benefit to have submitted to the Congress a paper on the subject Mr. McLauchlan had so ably treated. Anything that tended to enlarge the scope and sphere of actuarial science was of importance, inasmuch as, by this means, the status of the actuary received a wider recognition. He was glad that the question of censuses had been brought under the notice of the Congress. The first Actuarial Congress had performed a highly valuable work in securing the adoption of a uniform system of notation for the use of all countries. Similarly he hoped a future Congress would decide upon an uniform and universal system of classifying facts for census returns, which, as usually presented, were admittedly defective.

This paper was followed by one by Mr. St. John, of the Actuarial Society of America, on the intervention of the legislature for the purpose of permitting or supervising the operations of life assurance companies.

The paper of Mr. St. John gave an able account of the system of State supervision in the United States. It was followed by a note by Mr. Macaulay, of Canada, on the system in force in the Dominion. Both of the papers contained references to the standards of solvency attempted to be set up by various legislatures in America.

Mr. G. King gave the Congress some valuable information with regard to the law in England, and expressed his approval of the system of publicity and freedom under which we live, with regard to life assurance matters, while he thought, however, that

power should be given to the Government to stop the career of obviously insolvent societies.

Mr. McClintock stated that, in America, they had become accustomed to the system of State supervision, which had some considerable advantages. The principal inconvenience of the system was the want of uniformity in the laws of the various States of the Union, which caused an enormous amount of labour in order to comply with their manifold requirements. The evil of the system was that it fostered assessment and other schemes that were not amenable to the general laws on the subject of valuations.

The sittings of the Congress concluded with the proposal, by the President of the Institute of Actuaries, of a vote of thanks to the Organizing Committee, the President and Officials of the Congress, the Belgian Finance Minister, and the Towns of Brussels and Antwerp, who had extended their hospitality to the Congress generally. The motion was enthusiastically received, and carried by acclammation. The duty then fell to the President of the Institute to propose the place of meeting for the next Congress, which he then did, suggesting that it should take place in London, in May or June 1898. This proposal was unanimously welcomed and agreed to, without a dissentient.

M. Wolterbeek, of Amsterdam, supported the motions, but, at the same time gave expression to an opinion that the Congress had occupied itself too much with legislative questions, and too little with actuarial science.

M. Mahillon, the President of the Congress, defended their proceedings, by saying that they had not neglected the purely theoretical side of the questions before them. It would, however, be open to the organizers of the next Congress to arrange a programme in accordance with the views they might then hold.

I have now given, I trust, not too tedious an account of the business proceedings of the Congress, and hope I may venture to take up some further part of your time by a short account of the brighter side of the meeting.

The members of the Congress, from the day when they first met, until the time when they finally left Brussels, were received with the proverbial hospitality of the Belgian nation, and when it is stated that the entertainment of the Congress was organized and carried out as successfully as the scientific meetings, no greater idea can be given of the agreeable nature of the visit of the Congress to Brussels.

On the evening of Monday, the first day of the meeting of the Congress, the Bourgmestre and Echevins, of the Town of Brussels, received the members of the Congress, and the ladies who accompanied them, in the magnificent Hotel de Ville, where the whole of the splendid historical rooms were brilliantly lighted in their honour, and the ancient treasures of the building displayed to the greatest advantage. The delight of the visit was enhanced by charming music and bounteous hospitality.

On Tuesday evening, the Finance Minister and Madame Smet de Naeyer received the Congress at their handsome official residence, and entertained them in the most gracious manner.

On Wednesday, the President of the Congress and Madame Mahillon gave a magnificent banquet at their official residence to the Delegates sent to represent States or Societies at the Congress.

On Thursday, M. Le Jeune conducted a large number of the members of the Congress to Antwerp, where they were received by the Bourgmestre, who personally showed them the treasures of the Hotel de Ville. The visitors then proceeded to inspect the interior of the historic printing-house, founded in the 16th century, and now converted by the City of Antwerp into the Musée Plantain, where they were entertained most hospitably in the ancient Flemish courtyard of the building by M. Le Jeune. The company then returned to Brussels after having received an interesting souvenir of their visit printed on the ancient presses with the old types and on paper that has been carefully preserved for many years in the Museum.

On their return to Brussels, on the evening of Thursday, the English-speaking members of the Congress dined together. Dr. Sprague occupied the chair, and Mr. Shephard Homans acted as vice-chairman, while MM. Mahillon, Bégault, and Le Jeune were the invited guests of the evening.

On Friday, after the final meeting, a banquet was given to the Congress, when M. Mahillon, the President, occupied the chair, with the Belgian Finance Minister and the President of the Institute of Actuaries on his right and left, the posts of honour being accompanied by the customary penalty of speech-making.

The members of the Institute of Actuaries who were present at the Congress were unanimously of opinion that the representative capacity in which they attended secured for them the greatest honour and hospitality, and they left Brussels with the

conviction that the labours of the pioneers who founded the Institute of Actuaries, and the talents of those who succeeded them, had established a reputation for the Institute on the Continent of Europe of which they might be justly proud.

It rests with the present members of the Institute to sustain that reputation, and to endeavour to equal the hospitality shown at the Brussels meeting when the International Congress of Actuaries assembles in London in 1898.

Mr. George King, in proposing a vote of thanks to the President for his address, said the Institute owed to Mr. Finlaison a double debt of gratitude, this being the second address he had delivered during his term of office. He (Mr. King) could say from personal observation it added very much to the success of the Congress that Mr. Finlaison—in his double capacity of President of the Institute of Actuaries (which every one acknowledged to take the lead of all actuarial societies) and Actuary to the Commissioners for the Reduction of the National Debt—was one of the members of that gathering. From the succinct account which had just been given, one could judge of the large amount of work that was got through at the Congress. But one result that could scarcely be measured was the cordial relations which were established between the actuaries of many countries, to which result the presence of our President greatly conduced. A remarkable illustration of these cordial relations was the adoption of the notation of the Institute, which result could not have been obtained without personal contact with one another.

Mr. H. W. Manly, having seconded the vote of thanks to the President, which was unanimously accorded,

Mr. Finlaison, in reply, acknowledged the vote of thanks and hoped the notes he had put together would prove of value to the organizers of the next Congress. It remained only for him to adjourn the meeting to Monday, 30 December.

The Theory of Evolution applied to the System of Life Assurance and, incidentally, to Insurance generally.

[An Introductory Address delivered before the Birmingham Insurance Institute, on 1 November 1895, by T. E. Young, B.A., President.]

The subject of Insurance, theoretical and practical, has been so copiously and minutely discussed of late, especially in Lectures and Essays, that serious perplexity awaited me when I began to meditate upon my Introductory Address. I felt, under these

circumstances, that a Paper constructed on the usual lines must either weary you with tedious repetition or discredit myself with the reproach of platitudes. I accordingly determined to see whether I could not,—at all events, partially,—succeed in adding some novelty and interest to our proceedings by an attempt to ascertain whether the vast Doctrine of Evolution applied in any valid degree to the business in which we are engaged. Such an investigation possesses at least the merit,—however I may fail in my exposition,—of appealing to the sympathy and thought of all concerned in the several Departments of our common work, for results which may be found to bear upon Life Assurance will also, *mutatis mutandis,* apply to other Branches. I confess, moreover, that the subject strongly attracted my attention on the ground that an enquiry of this interesting character had not previously been attempted.

It might be justly assumed *à priori* that, inasmuch as Insurance is the objective, concrete expression of certain subjective needs and desires of human Beings, combined into a Community, a Theory which is demonstrably true of those Beings must also be latent, and capable of explicit exhibition, in the feelings which they have embodied in the Scheme of Insurance. And I accordingly now proceed to describe the essential elements of this luminous doctrine prior to a detailed application to our work.

The concurrent discovery of Mr. Darwin and Mr. Wallace is often popularly confounded with the entire Theory of Evolution. Their hypothesis, however, was restricted to an explanation,—fertile in suggestiveness and approximately complete—of organic succession in the vegetable and animal Kingdoms,—based upon the factors of spontaneous variety of individual characteristics; heredity of special traits; and, by reason of the stress of excessive multiplication, a struggle for existence resulting in the Survival of the Fittest in adaptation to the physical Environment. This, however, is only one form of application of Evolution, and, for a graphic and marvellous delineation of the entire range of the Method,—starting from the incandescent or gravitational origin of the Universe, and terminating in the development of Man and his capacities and attainments,—physical, moral, and intellectual, we must turn to the massive and monumental intellectual edifice which has been reared by the constructive and all-embracing genius of Mr. Herbert Spencer. Throughout the whole of his separate Essays and in his memorable Synthetic Philosophy he

has consistently and organically expounded and traced the connected process of Evolution, for which he has been fitted both by an original and commanding genius and by a sedulous intellectual training and universal scientific and philosophic equipment, which amply justify the admiring comment of Mr. George Henry Lewes in his History of Philosophy, that "It is "questionable whether any thinker of finer calibre has appeared "in our country." To his works, therefore, I have recourse in my preliminary statement of the prominent features of the Evolutionary process.

Our first step, in accordance with established usage, is to define our Terms. Evolution, etymologically considered, means a process of unfolding or gradual development; the process by which a simple body successively assumes a more and more complex and highly organized condition; not by mere mechanical juxtaposition of added elements, but by an internal growth and differentiation, where, in response to varied external forces and changing environment of physical conditions, the original body, by its inherent vitality and adjustive powers, adapts itself from stage to stage. In this process we observe two facts; (i) That the ultimate complexity of structure and final assemblage of activities and functions are really latent in the original formless germ, and constitute merely a development, and (ii) That the progress in differentiation, like all natural processes, is gradual and continuous, and not *per saltum.*

We note, in passing, that the more complex the structure becomes, the greater necessarily is the number of relations which, in its ascent, it establishes with the environment, and the larger, consequently, becomes the multitude of points and opportunities which surrounding antagonistic forces can seize for assigning to Dissolution, or the process of disintegration, a supremacy over the constructive process of Evolution. We perceive an example of Evolution in the scientific divination of the poet Göthe that the complicated structure called a plant, is simply an unfolding or transformation of the stem and leaves; while the exquisite and variegated animal Kingdom is but the developed expression of a primitive structureless protoplasm.

One or two further definitions will suffice to introduce us to the intelligent understanding of the formula of Evolution at which Mr. Spencer finally arrives. The phrase he employs of "the Dissipation of Motion and Integration of Matter" means that the separations and distances and repulsions existing at any

time between bodies, and between the molecules of bodies, gradually diminish, with the consequent tendency to aggregation, or the formation of wholes; while the counterphrase of the "Absorption of Motion and consequent Disintegration of Matter" involves the precisely reverse process. At the risk of tediousness, but with a view to clearness of conception, let me add that the "Dissipation of Motion"; its gradual reduction; brings together, or to nearer neighbourhood, bodies that were moving away from each other on divergent paths; thus confining them into one locality: and similarly with the constituent particles of bodies, and their closer contiguity to one another, with the consequent formation of denser masses.

So the term "coherent" implies the more or less decided and definite compacting of bodies, and the parts of bodies, in orderly form, and this not a mere mechanical contiguity but an organic unity; while, on the other hand, the word "incoherent" signifies the opposed result. Again, by "homogeneity" is intended the likeness in nature and structure of each portion of a body to the rest of the constituent parts; while "heterogeneity" indicates a diversity of character in the several elements of the body and their sequent functions.

We are thus prepared, by this brief recital, to follow intelligently Mr. Spencer's formula, and to perceive its adaptability as explanatory of the origin of the Insurance system and its diversities of form.

Mr. Spencer first ascertains that, throughout the Universe, the complete series of changes in all sensible existences, whether individual or social, during the ascending period of their history, (with which I deal alone in this Address), consists of the dissipation, or loss, of the motion which keeps bodies, and parts of bodies, asunder, and their consequent integration or compacting into wholes.

Thus, by abolition of differences or separations, we obtain, in the social world, the integration of Kingdom into Kingdom, and the coalescence of contiguous Kingdoms into a single Nation. Towns become integrated by growth through the absorption of suburbs, the distance between the original town and suburb, generalized as a form of motion, disappearing with the growth of population and the community of similar interests. The practical absorption by Staffordshire of the Pottery Manufacture, and the consequent decay of this industry in Derby and elsewhere, is another example of integration through the monopolization of

a special trade by reason of exceptional local advantages. In Commerce we perceive the process of integration proceeding in the establishment of centres of trade, such as the Bankers' Clearing House, and the several Exchanges for different descriptions of commodities. Similar businesses thus naturally tend to aggregate to a common locality, by dissipation of original divergent movements of the workers in the industry. This process is clearly one of a passage from incoherency to coherency; a transfer from a condition in which the several units are separate to one in which they constitute an organic and interdependent group. Thus, in its primary aspect, Evolution is a change from a less coherent to a more coherent form, consequent on the operation of the principle of dissipation of motion and integration of matter. But Evolution exhibits a second aspect: like existences are integrated or aggregrated into wholes, but the wholes divide themselves into functional parts: the parts again present a process of differentiation, and hence we finally arrive at a complex organization.

This passage from the Simple to the Complex is characteristic of all growth: and the German Physiologist, Von Bär, had long previously generalized the observed phenomena into the universal formula that each organism proceeds from a homogeneous state to a condition of heterogeneity. Consider how the race of man has become differentiated, through the influences of soil, climate, and varied environment, into multiplied races, profoundly differing from each other in their specific characters and organization. Notice, again, how, in the same race, a complexity gradually emerges: distinct functions in the community are adopted by different sections of the members: a special condition of one part of the country inhabited converts one set of toilers into agriculturists; the presence of coal in a neighbourhood produces the specialized industry of mining; the existence of abundant clay and facilities for working it again differentiate the labourers into Potters; and, as a striking example of differentiation in a single industry, which you, who have read the interesting pages of Adam Smith, will remember, I may cite the complex division of one single branch,—that of pin manufacture,—into a number of separate processes and their workers.

The generalization of Evolution also includes the changes from the Indefinite to the Definite: an advance from confusion to order: to determined arrangements; and hence, proceeding in this gradual fashion, we at last reach the completely generalized

formula — that Evolution is an Integration of Matter and concurrent Dissipation of Motion, during which the Matter passes from an indefinite incoherent homogeneity to a definite coherent heterogeneity.

Though expressed in technical philosophical language, you will now perceive, from the explanations which have preceded, the truth and validity of this doctrine. And, moreover, bear in mind that, obtained inductively as the generalization has been from a survey of organic and inorganic history by the keen and disciplined mind of a Master, the formula sums up both the phenomena of the past and of the future, and involves the diversified manifestations of individual and social life. I need simply add that, concomitant with this process and included within it, we trace the uniform fact that all motion and growth tend to travel along the line of the greatest attraction, or the line of least resistance, or the resultant of the two. I need only briefly recall to your memory one or two of the illustrations I have already employed, as concrete examples of this natural progress.

I now approach the more difficult but, at the same time, the more interesting portion of my task, in the endeavour to point out the exhibition of the formula which the history of Assurance displays.

I may begin by showing that the history of Insurance is a passage from the homogeneous to the heterogeneous as a portion of the general development of business enterprise. A merchant, originally, in the less complex conditions of commerce, would run the risk himself of Fire or Marine disaster, and the losses would be simply a reduction of his Capital resources. As business became generally more complex and heterogeneous with the growth of population; the colonization of waste places of the world; the stress of competition; the opening of new markets; and the increase of commercial enterprise, the possible risks would become too grave for an individual to incur, with the serious chance of permanently crippling the power of industrial expansion; and hence the specialized system of Fire and Marine Insurance was naturally created to the invaluable relief, by adoption of the basis of averages, of a merchant's resources and capacity of adventurous trade. The independent method of Insurance was thus a fragment or department of risk broken off and segregated from the general body of business, as that business expanded from the Simple to the Complex under the force of advancing civilization. Some Shipping Companies even now possess an

Insurance Fund—a remnant of ancient practice—which provides for the risk of losses at sea.

As a matter of history, it may be mentioned that, as would naturally be conjectured, the principles of Insurance were chronologically first applied to Marine casualties, and subsequently to risks from fire.

Again, in illustration of the natural tendency of things to advance along the line of least resistance, we might pause for a moment to compare the progress of British and American Assurance Offices. It cannot be affirmed that the wealth, solidity, and number of British Companies, as compared with those in America, are due to any deficiency of mercantile enterprise among the American people, but are largely dependent on the fact that advancement in the one case is prevented by the unnatural obstruction interposed by the conjoint forces of Commercial Protection and minute Government intervention, while in the other case the line of least resistance is kept clear and free by the invigorating force of Freedom of Trade and of that ample scope for development which combined enterprise possesses wherever the enfeebling and paralyzing power of Government control is unknown.

We see the prevailing and primary principle of Dissipation of Motion and concurrent Integration of Matter—the gradual massing together of bodies by diminution of the intensity of forces producing separating movements—in (i.) the fact that Assurance Companies have more and more become congregated in the City (of London) and in specific localities. In the City of London itself, we find collected 22 Head Offices and 21 Branches of the leading Institutions, while the Class Offices,—such as the Law Offices—find a common centre about the Law Courts and the chambers of lawyers. In former times, specific companies were established in certain provincial towns as designed to meet the Assurance requirements of various Counties and portions of the Kingdom. I may mention the "West of England" at Exeter; the "Provincial" at Wrexham; the "United Kent" at Maidstone; the "Friends' Provident" at Bradford; the "Yorkshire" at York; the "Midland Counties" at Lincoln; and several others. Though some still maintain their provincial position, the constantly operating principle of dissipation and integration has, of recent years, asserted itself more effectively, and now the "West of England", the "Provincial", and the "Midland Counties" have become, by transfer and amalgamation,

absorbed in larger offices in London as a centre. Marine Companies tend to aggregate in proximity to Lloyds' and the offices of the Brokers and Merchants. And the general tendency to consolidation appears likely to continue in embodiment of this universal principle. Another illustration (ii.) of the existence of this predominant process consists in the larger and more important amalgamations of Companies which have occasionally occurred, when two offices lose their separate motions (so to speak), or their independent existence, and coalesce into a denser aggregate. As an additional piece of evidence (iii.) of this tendency to integration, we find that the separate Assurance bodies still further seek,—in conformity with this natural law,—to aggregate themselves into closer union through the establishment of the Institute of Actuaries and the Life Offices' Association. In the former Institution, we have an Association of professional men, endeavouring to introduce scientific order and rationality into the technical and expert conduct of the business; and a reference to the 31 volumes of its published Journal, and to the scientific methods which now, through the influence and authority of the Institute, generally prevail, affords an impressive testimony to the enduring value and importance of its labours. In the Life Offices' Association, we perceive, not an association of individual experts only, but a combination of Companies attempting to guide and render uniform, without coercion, the practical character and direction of Life Assurance work, and destined, I believe, to afford signal service to the Companies themselves and consequently to the assuring public. The Tariff Association is a similar body in relation to Fire Offices; and the Institute of London Under-writers sustains a relation to Marine Companies corresponding to both the Institute of Actuaries and the Life Offices' Association.

I need not pause to point out that the system of Re-assurance, which prevails among Companies, is another example of the progressive integration of which I have been adducing illustrations. This device not simply binds offices together into a more coherent system, but possesses the additional advantage of producing coherency and uniformity of practice between them.

In further pursuance of this branch of the subject, it is pleasant to refer to the formation of various Insurance Institutions throughout the Kingdom as evidence of an effort after integration and consolidation: and it is sincerely to be hoped that these Bodies may become more and more intimately and organically

allied with the Institute of Actuaries and the Life Offices' Association of London. Indeed, I may venture humbly to assert that, besides the honour and pleasure of becoming personally connected with the Birmingham Institute, one main object I had in view in accepting the Presidency was, as far as my efforts and influence could extend, to aid in cementing this closer union between the provincial Institutions and the central authorities.

Passing now to the change from homogeneity, or general likeness, to heterogeneity, or variety of feature, I may point out that, speaking generally, Assurance business originally consisted mainly of the three principal departments of Fire, Life, and Marine, or, placing them in chronological sequence, Marine, Fire, and Life. At the present time, however, the heterogeneity of the business is exhibited by the additional functions,—necessitated by the expanding complexity of commercial relations—of Insurance against Breaches of Trust; Burglary; Hailstorms; Accidents; damage to Plate glass; destruction of Cattle; provision in Sickness; redemption of leasehold investments; Insurance to render Interests marketable which would be defeated by the birth of Issue; and Insurance to compensate the diversion of properties which would occur if a restrictive "Name and Arms" clause in a Settlement or Devise were violated. While in Germany and other Continental Countries, as you are aware, the complex scheme now embraces Insurance in respect of Old Age. Besides this varied aspect of the Insurance System generally, we trace the same advance from homogeneity to heterogeneity—from sameness to diversity—in the circumstance that individual Companies now comprise, in addition to Fire and Life Insurance, or even Fire, Life, and Marine, the application of Insurance to Leaseholds, Issue risks, "Name and Arms" protection; with powers in some cases to include Fidelity and Accident Contingencies. A further illustration of the change from the homogeneous condition to one of heterogeneity is afforded by the fact that Companies originally consisted of a Head office with a few scattered agencies, while we now perceive an elaborate and complex system of Branches and Agencies throughout the world, with local and general Inspectorships connecting them together.

I might also direct your attention to the fact, as a part of this specific subject, that formerly Class offices existed of a practically homogeneous character, which chiefly transacted a definite section

of Insurance work, such as the "Colonial", which devoted itself mainly to Assurances upon persons resident in the Colonies and India or proceeding thither; Companies established primarily for a special class of Professional men, such as the "Engineers'"; Companies originally designed especially for the Assurance of Members of the Universities, or of clergymen of the Church of England; and Companies founded mainly to meet the needs of various Dissenting Communities. These homogeneous Companies, however, exhibit the principle of advance to heterogeneity, for either they have enlarged their boundaries to comprise a wider area of business, or they have become merged (like the "Colonial") into other offices showing a larger variety of scope.

The progress to a heterogeneous condition is also evidenced by the historical fact that combinations for Life Assurance were originally of a Mutual character; for obvious reasons Proprietary Companies were next established, in which the assured were protected by a subscribed capital, while the whole of the Profits belonged to the Shareholders; and, finally, there was reached, in further heterogeneity, the prevailing form in which, with the special security afforded by the second type of Company, the Assured were admitted to a definite share of the Profits.

I might also adduce the fact that, originally, the Assured in a Life Office were charged a uniform premium without reference to age; and that, although the old "Amicable" was established in 1706, it was only in 1807 that its members secured the right by Charter to graduate their rates of premium according to the mortality prevailing at each age. Up to the latter epoch, all Life contributions, as I have stated, were practically the same for every age.

The examples I have furnished in treating the subject of integration involve also, without the necessity of detailed exposition, the advance from an Incoherent to a Coherent condition. In this aspect of the question we perceive a gradual development from a more diffused to a more consolidated state; an increase of combination among the individuated portions of a whole; an advance, towards greater concentration, of elements which were previously scattered, with the result that these several elements tend to become mutually dependent, and hence to constitute,—in the scientific sense of the expression,—an organic system. I have already exemplified the process, outside the specific application of this Address, by indicating the gradual

coalescence of neighbouring Kingdoms, in the course of National history, into a single Empire. Applying the principle to our present subject we note the passage from an incoherent to a coherent condition in the aggregation of Companies through the instrumentality of the Institute of Actuaries and the Life Offices' Association,—both Institutions being expressions of this principle; in the system of Reassurances, which connects Companies together into a more intimate and interdependent unity; in the practice of Amalgamations; and in the general employment of one public and well-known Table of Mortality, based upon appropriate and general data, for the construction of premiums and the formation of Reserves, in place of a multitude of Tables founded upon partial statistics.

It may also be fairly stated that the Life Assurance Companies' Acts of 1870 and 1872 constituted an epoch in this natural tendency to a coherent condition.

Another element in our general formula of Evolution was the change from an Indefinite to a Definite state: a progress from confusion to order; an advance from indeterminate arrangements to those of a determined character, with the consequence of closer precision and more organic structure in our systems. Illustrations of the existence of this tendency in Life Assurance are readily discerned. In former periods of the history of the business, no thought appears to have been bestowed upon the equitable necessity of a relation between the premiums charged and the methods adopted for the distribution of surplus. This serious flaw in our plans is now being rectified. In earlier days, the premiums actually charged were deduced from a Table of Mortality, usually the "Northampton" Table, without any addition,—technically, though infelicitously, termed "loading"—and the expenses and profits were assumed to be provided from a favourable mortality experience and an excess in the rate of interest realized beyond that involved in the Valuation. At the present time, we seek more scientifically to deduce from suitable data the premium for the risk of death alone,—usually termed the "pure" premium,—and additions are then made, in various forms adapted to the requirements involved, for the expenses of management, the possible advent of fluctuations of experience, and the creation of Bonus funds. In Valuations, again, we adopt preciser and more adequate modes than those formerly in universal use. We do not, as in the primitive times, when the "Northampton" Table reigned supreme, adopt a homogeneous

method, and deduce our values of Annuities and Reversions from the premiums actually charged; but, as an advance to heterogeneity, we employ an appropriate Table of Mortality for the Annuities and Reversions by which the Reserve for liabilities, compounded of the values of the future pure premiums and of the sums assured, is to be formed; though we obviously ascertain the present value of the annual excess in the office premiums beyond the pure premiums, and retain it specifically for the purposes for which it was imposed: and various modifications are introduced in the form of additional reserves for the payment of claims at an earlier date than the Tables of functions provide; for the unequal distribution throughout the year of the Premium income; and for the retention, in equalization over the currency of the contracts, of a sufficient portion of the "loading" included in premiums which are limited to a specific number of payments. On this subject, another fact of Assurance history is interesting and relevant. In the early stages of Life Assurance, a fixed sum payable at death was unknown; the members who died at any time shared proportionately in the existing funds of the Company; and it has been shown that, prior to 1760, when discussions occurred preliminary to the foundation of the old "Equitable" in 1762, no plan had ever been formed for the assurance of a fixed amount at death whenever it might take place in consideration of a premium continuing from the date of the contract to the termination of life.

An interesting suggestion may now be made. As far as I can gather from the history of Life Assurance, the earliest Policies (those of the "Amicable", long ago transferred to the "Norwich Union") were assurances effected for terms of years; and Mr. Frederick Hendriks has pointed out that, as is proved by an Affidavit made in 1761 on behalf of the "Royal Exchange" Corporation in opposition to the grant of a charter to the "Equitable", the business of the former, during the 40 years which had elapsed since the date of its establishment, had almost exclusively consisted of Policies for short periods only, *i.e.*, for a year or for a lesser term. Now, in the Darwinian Theory, a principle is elucidated of the Reversion to primitive and discarded forms of life; and Mr. Darwin has lucidly traced its operation both in the vegetal and animal Kingdoms. A structure which, through a careful process of selection, has become highly specialized and

developed from an inferior form, will, under certain circumstances, tend to revert to its ancestral and more rudimentary type. Now the vaunted modes of Assurance which have appeared under the names of the "Natural Premium" plan and the "Assessment" system are essentially a series of Term Policies, with the premiums ascending age by age in accordance with the increasing rate of mortality, in substitution of the modern and more developed system of a uniform or "level" premium throughout life. Hence these spurious schemes are really examples, in the Commercial Kingdom, of the principle of Reversion, and show that the propounders are attempting to entice the public into primitive methods of Assurance which the rational elaboration of the business has demonstrated to belong to the incipient and imperfect stages of its history.

I have now completed the task which I assigned to myself, and, although the exposition has been, I confess, somewhat abstruse and difficult, and certainly, I regretfully admit, of a fragmentary and imperfect character, yet I deemed it worth attempting on account of its deeply interesting nature as ranging the phenomena of Insurance under the sway of the general evolutionary principles which Mr. Spencer has expounded, and more particularly, on the further ground that no investigation in this direction had hitherto been essayed.

I have purposely dealt only with the prior portion of universal evolutionary history—that in which the subject proceeds during the ascending period of its natural course: for, crowning and consummating the general process, there yet remain the consideration of the instability to which all advanced and intricate structures are exposed, and, as a complete exposition of the doctrine, the reverse process of Dissolution. But this concluding portion of the universal process is, happily, of too purely speculative, remote, and visionary a character to be applied to our subject. When the human race and its economic relations have attained their perfect realization, some successor to myself may, in that more fortunate era of the absence of contingencies, dilate upon the completed history, and mark and map the correlated march of Dissolution; but we are meantime restricted to the profitable duty of tracing the beneficent history of Insurance in the past; of observing the succession of its natural stages, and its amplitude and flexibility of fertile adaptation to the advancing and varying needs of man and his social arrangements; and of

predicting that, momentous and impressive as have been the applications of Insurance in the past, it is destined in the future to confer an ever-expanding benefit upon the social conditions of life, and to aid materially in promoting a completer co-operation of men in sympathetic action, with a consequent closer communion between the nations of the Earth.

A Method of approximating to the Effect of a Reduction in the Rate of Interest used in calculating Reserves under Whole-Life Policies; with Tables.

By JOHN STEWART, F.F.A., F.I.A., *Assistant Actuary of the City of Glasgow Life Assurance Company.*

IN view of the present movement on the part of life assurance companies towards a reduction in the rate of interest used in calculating their reserves, it is desirable to have some easily applied method of measuring approximately, with sufficient accuracy for the purpose of a preliminary estimate, or a rough check on fully computed valuation figures, the effect of such a change. The process is simple as regards whole-life policies, classified according to age attained, if the premiums valued represent some stated proportions of the gross premiums; but difficulty arises when the premiums used vary with the rate of interest, as in the case of a net-premium valuation. The following Tables I to VI (see pages 268–271) supply the means of approximating from the net-premium reserves for whole-life policies, under the H^M or combined H^M and $H^{M(5)}$ Tables, at most rates of interest presently used, to the corresponding reserves at rates one-half or one-quarter per-cent less.

PRINCIPLE, AND PRACTICAL USE OF TABLES.

If i represent a higher rate of interest than j—an assumption which it will be convenient to make in the further use herein of these symbols—the ratio $\frac{V_{(j)}}{V_{(i)}}$, at any age attained, increases with the duration of the assurance. This is shown by the following table, giving the H^M values of

$$\left(\frac{V'_{(2\frac{1}{2})}}{V'_{(3)}}-1\right)\times 100,$$

where $V'=V+\frac{P}{2}$—the function which will be found most suitable for the purpose in view:

Age at Entry	Age attained		
	40	50	60
20	8·70	6·62	4·67
25	8·40	6·43	4·55
30	8·15	6·27	4·45
35	7·77	6·08	4·34
40	...	5·86	4·22
45	...	5·61	4·11
50	...	...	3·98
55	...	...	3·82

These figures suggest that the assumption of a uniform ratio existing at each age attained, irrespective of age at entry, between the values of policies at different rates of interest, would afford a very rough means of making the desired approximation. A much better result, however, may be obtained by operating, not directly upon the policy-value, but upon the separate values of sum assured and future premiums, and applying a slightly higher ratio of increase (the difference being nearly constant) to the latter value than to the former—thus introducing to the calculations an element depending altogether on the value of the premiums, the only function of the duration of the assurance practically available. The following is a comparison of the H^M $2\frac{1}{2}$ per-cent values of $\left(V+\frac{P}{2}\right)\times 100$ with the approximate values of the same expression arrived at by making the percentage additions shown at the head of each column to the H^M 3 per-cent values of sum assured and future premiums respectively—

Age at Entry	Age attained								
	40			50			60		
	Increase in Values of Sum Assured . 8·65 % Future Prems. 8·80 %			Increase in Values of Sum Assured . 6·5 % Future Prems. . 6·7 %			Increase in Values of Sum Assured . 4·5 % Future Prems. . 4·7 %		
	True Values	Approximate Values	Difference	True Values	Approximate Values	Difference	True Values	Approximate Values	Difference
20	23·732	23·682	−·050	38·454	38·369	−·085	54·378	54·258	−·120
25	19·875	19·879	+·004	35·354	35·331	−·023	52·096	52·035	−·061
30	14·966	14·985	+·019	31·408	31·422	+·014	49·192	49·175	−·017
35	8·944	8·959	+·015	26·568	26·610	+·042	45·629	45·653	+·024
40	...	...	...	20·473	20·523	+·050	41·143	41·199	+·056
45	...	...	...	12·447	12·462	+·015	35·235	35·300	+·065
50	...	...	...	...	...		27·526	27·581	+·055
55	...	...	...		...	...	17·183	17·193	+·010

The figures in the accompanying tables are intended for use on the lines above indicated, and the method of their application is obvious. It need hardly be stated that they do not in any case produce the value of sum assured or future premiums, but quantities, meaningless in themselves, whose difference forms an approximation to the policy-value.

As in practice the values of assurances are not generally taken out for each age attained, the values of sums assured and future premiums, as given in the office valuation books, would in any similar method of approximation form the material to be operated upon; and no increase of labour is caused by the ratios applicable to those respective values at any one age differing slightly from each other.

In most cases an approximation arrived at by collecting into suitable groups the values of sums assured and future premiums at the higher rate of interest, and using average or central multipliers, would probably be sufficiently accurate. The work involved in the application of the individual ratios, however, should not occupy more than two or three hours. As the number of multiplying figures is generally only two or three, and the second decimal figure is either 5 or 0, the results, to the nearest integer, of the multiplication by the separate figures of the ratios may easily be set down at once in columnar form, and the total addition to the values of the sums assured or future premiums at the higher rate of interest ascertained by one summation.

DEGREE OF ERROR INVOLVED IN THE APPROXIMATION.

Tables I *and* II—The greatest difference, under these tables, between the true and approximate values of policies of more than five years' duration, on the lives of entrants at all ages from 20 to 70 inclusive, is ·3 per-cent of the correct value. The average error will probably be considerably less than one-half of this figure, which only occurs once or twice, and is the greatest of the series of maximum errors relating to the different ages attained. The errors of excess and deficiency will partly cancel each other, and it is believed that, except in the case of a company with a very abnormal distribution of its risks over the various ages and durations, this will take place to a large extent, leaving only a small balance of error.

If the unfavourable assumptions be made that in any particular case the average error will amount to ·15 per-cent of the correct values, and the sum of the positive errors to four

times that of the negative errors, the total approximate valuation of this part of the business will be incorrect to the extent of ·09 per-cent of the true reserves (this figure being reduced or over-balanced by the error in the approximate valuation of the newer policies).

The error under assurances of from one to five years' duration is greater, when measured as a proportion of the values, than under older policies, this being, however, counterbalanced by the fact that the values of new assurances are very small. The maximum error per £100 assured is, for ages at entry 20 to 45, ·035; and for ages at entry 45 to 70, ·05. The result of the divergence from the true values will here be on the side of deficiency, to be set against the error in the approximate valuation of the older policies, which will generally be on the side of excess. A large error with the same sign—positive or negative—under each section of the business can hardly arise.

If it be assumed that the average error under policies from one to five years old will amount to ·02 per £100 assured, and the sum of the negative errors to six times that of the positive errors, the net difference between the true and approximate valuations of this part of the business will be ·0143 per £100 assured—or, say, £715, if the existing assurances effected during the five years preceding the valuation amount to £5,000,000.

As Tables I and II give better approximations to the true values in individual cases than Table V, it may be inferred that the test hereafter applied to the latter table (see next section) would show still more satisfactory results if either of the former tables were used.

Tables IV *and* V.—The ratios of these tables fall to be applied to the H^M values of new assurances, as well as to the values of older policies by the combined H^M and $H^{M(5)}$ functions, and the errors in individual cases are consequently larger than those occurring under Tables I and II, although the differences are not practically great. It was therefore deemed expedient, after framing Tables IV and V, to apply some independent test of their working as a whole; and Table V was chosen for this purpose, as the figures used in its formation had to be specially computed, and the tests otherwise applied to it were not so full as those to which Table IV was subjected. The following experiment may, however, be regarded as referring, *mutatis mutandis,* also to Table IV, as both tables were framed on exactly the same lines.

In a paper "On the approximate calculation of Valuation Reserves", printed in vol. xxvii of the *Journal of the Institute of Actuaries*, Mr. Jas. Chisholm gives a table (page 446) showing the sums assured with a particular company, classified according to quinquennial average ages at entry, and quinquennial periods of duration. To avoid, as far as practicable, the use of values at fractional ages, the figures of that table were, for the purpose now in view, redistributed, as regards duration, into the following periods—"0-5" (as given by Mr. Chisholm)—"5–7½" (= one-half of amount in period "5–10" of table); "7½–12½" (= mean of respective amounts in periods "5–10" and "10–15" of table); "12½–17½", &c. The sums so obtained were then multiplied by the true $H^{M(5)}$ 2½ per-cent values of $\left(V+\frac{P}{2}\right)$ for the central ages at entry and valuation, and also by the corresponding approximate values given by the use of Table V (before the ratios in that table were amended as hereafter mentioned).

The results were—

Amount of true $H^{M(5)}$ 2½ per-cent Values	. .	£571,696
" approximate " "	. .	571,590
Difference	. . . .	£ 106

To obtain some idea of the effect of changes in the distribution of policies over the different ages, the figures were divided into three groups, according to duration, and combinations were made of the amounts of these groups, singly and in pairs, with those of the other group or groups, multiplied by five. The following table gives the results:

Duration	True Values	Approximate Values	Difference
	£	£	£
A – 0 to 17½ . .	131,815	131,836	+ 21
B – 17½ to 37½ .	339,564	339,519	– 45
C – 37½ and upward	100,317	100,235	– 82
5A + B + C . .	1,098,956	1,098,934	– 22
A + 5B + C . .	1,929,952	1,929,666	–286
A + B + 5C . .	972,964	972,530	–434
5(A + B) + C . .	2,457,212	2,457,010	–202
A + 5(B + C) . .	2,331,220	2,330,606	–614
B + 5(A + C) . .	1,500,224	1,499,874	–350

These figures indicate that Table V would give the desired approximation with abundant accuracy under widely differing circumstances as regards incidence of ages. They suggest,

however, that the error would in most cases be on the side of deficiency; and as a careful examination confirmed the suspicion thus raised, the table ratios were slightly adjusted to meet this defect—the new figures being those given in Table V. The other tables were also examined with this point in view, and such amendment made in the figures as seemed desirable. It may be mentioned that the alterations thus effected, giving roughly some weight to the probable incidence of the ages at entry, have made the errors in individual cases greater than they would otherwise have been, and have interfered slightly with the symmetry of the tables. The amended figures will, however, bring out better total results.

Tables III *and* VI—The errors in the approximate values given by the use of these tables may be stated roughly as rather more than one-half of those arising under the other tables at the corresponding rates of mortality.

Adjustments.

Reversionary Bonuses—A separate estimate will, of course, require to be made of the value of existing reversionary bonuses.

Incidence of Premium Income.—The values to which the ratios of increase given in the tables refer are those represented by the expression $V+\frac{P}{2}$—*i.e.*, the next due premium is assumed to be payable six months after the valuation date. If the interval be taken generally as m months, the formula becomes $\left(V+\frac{P}{2}\right)+P\left(\frac{m-6}{12}\right)$, and as the proportional changes in the values of P consequent on reductions of the rate of interest are less than the ratios given in the tables, some adjustment will be necessary if m be not equal to 6. The amount of this correction may be taken as

$$-\text{(average excess of table ratios over ratios relating to the values of P)} \times \frac{m-6}{12}$$

$$\times \frac{\text{total net premium income at interest } i}{100},$$

where the expression within brackets represents the following quantities:

Under Table I 3·95,	Under Table IV 3·33,
" " II 3·81,	" " V 3·16,
" " III 2·06, 2·00, 1·94, or 1·88.	" " VI 1·71, 1·65, 1·59 or 1·53,

the quantities under Tables III and VI varying according as the rate of interest to be reduced by one-quarter per-cent is $3\frac{1}{2}$ per-cent, $3\frac{1}{4}$ per-cent, 3 per-cent or $2\frac{3}{4}$ per-cent.

These figures are arrived at by multiplying the number of entrants at quinquennial groups of ages, according to the Institute of Actuaries' Life Companies' experience (taken from the table given by Mr. R. R. Tilt in vol. xxxii of the *Journal of the Institute*, page 6) by the differences, at appropriate central ages, between the ratios referring respectively to $\left(V+\frac{P}{2}\right)$ and P. A little examination will show that this approximation is sufficiently accurate for practical purposes. The greatest difference under any of the tables, between the average figure given above, and the actual differences between the two sets of ratios at quinquennial ages from 20 to 45 inclusive, is ·6. In the recently issued valuation report of one of the largest British assurance companies, the net premium income under whole-life policies is given as £454,004; and if m be taken equal to 8, and it be assumed that the true average difference between the two sets of ratios differs in this instance from the approximation given above by, say, ·3, the total error thus arising will, in the case of the company referred to, amount to £227, on reserves of about £8,000,000.

Ages at Valuation.—The figures given in the tables are those for exact ages. For fractional ages, interpolated values should be used—or, for mean ages, the ratios at ages alternately above and below the valuation ages may be taken.

Paid-up Policies.—The values of $\left(\frac{A_{j}}{A_{(i)}}-1\right)\times 100$ are greater than the figures given in the Tables—the difference amounting at age 30 to about 4 or 2, according as the proposed reduction in the rate of interest is $\frac{1}{2}$ or $\frac{1}{4}$ per-cent, and decreasing in either case to a vanishing point at the older ages. An adjustment may therefore be found necessary should policies on which no future premiums are payable be included with the ordinary policies in the valuation lists. If it be considered that the proportion of paid-up policies so treated is large enough to call for a correction, the necessary addition to the sum brought out by the use of the tables, though somewhat more troublesome to compute than the figures of the other adjustments referred to, may be arrived at without any great labour by collecting in a few age-groups the amounts of the paid-up policies, extracted from the valuation books, and applying the values of A at central ages, and the

differences, at corresponding ages, between the ratios referring respectively to A and $\left(V+\frac{P}{2}\right)$.

Bonus Reductions of Premiums.—The use of the tables in cases where bonuses have been applied to the reduction of premiums assumes that the ratios given therein refer to the values of $\left(V+\frac{P}{2}\right)+r\ (\cdot5+a)$, where r represents the amount of bonus reduction per unit assured. The error caused by this assumption at age at valuation $(x+n)$ may be expressed thus:

$$\left[\frac{\text{Table ratio for premium values at age } x+n}{100}-\left(\frac{\cdot5+a_{x+n}(j)}{\cdot5+a_{x+n}(i)}-1\right)\right]$$

$\times$ value at interest i of bonus reductions under policies valued at age $(x+n)$.

From the following table, comparing the ratios for premium values of Table IV with the $H^{M(5)}$ values of

$$\left(\frac{\cdot5+a_{(3)}}{\cdot5+a_{(3\frac{1}{2})}}-1\right)\times 100.$$

it will be seen that, where a special feature is not made of the distribution of bonuses by the method under notice, the disturbance arising from this source may be disregarded—more particularly when it is considered that the amount of bonus reduction increases with the age, and that at the older ages the differences between the two sets of ratios are insignificant.

Age at Valuation	Ratios of Table IV	Ratios referring to $\cdot5+a$	Difference
40	8·55	6·40	2·15
50	6·50	5·14	1·36
60	4·70	3·80	·90
70	3·00	2·70	·30
80	1·75	1·70	·05

In the case of the company mentioned by Mr. Chisholm in his paper already referred to, the total value of bonus reductions of premiums is stated as £4,000. If the average difference between the above two sets of ratios be taken as 1, the total error caused by bonus reductions will in this instance amount to £40, under reserves of £529,757.

Age at Valuation	Table I		Table II		Table III	
	H^M Mortality—Rate of Interest to be altered from					
	$3\frac{1}{2}\%$ to 3%		3% to $2\frac{1}{2}\%$		$3\frac{1}{2}\%$ to $3\frac{1}{4}\%$—$3\frac{1}{4}\%$ to 3% 3% to $2\frac{3}{4}\%$—$2\frac{3}{4}\%$ to $2\frac{1}{2}\%$	
	Increase per-cent in Values of					
	Sums Assured	Future Premiums	Sums Assured	Future Premiums	Sums Assured	Future Premiums
20 & under	11·15	11·20	11·35	11·40	5·60	5·65
21	11·20	11·25	11·40	11·45	5·65	5·70
22	11·25	11·30	11·45	11·50	5·70	5·75
23	11·30	11·35	11·50	11·55	5·75	5·80
24	11·35	11·40	11·55	11·60	5·80	5·85
25	11·40	11·45	11·60	11·65	5·85	5·90
26	11·20	11·25	11·40	11·45	5·70	5·75
27	11·05	11·10	11·20	11·25	5·55	5·60
28	10·95	11·05	11·10	11·20	5·45	5·50
29	10·75	10·85	10·90	11·00	5·30	5·35
30	10·55	10·65	10·70	10·80	5·15	5·20
31	10·35	10·45	10·50	10·60	5·10	5·15
32	10·15	10·25	10·30	10·40	5·00	5·05
33	10·05	10·20	10·15	10·30	5·00	5·10
34	9·90	10·05	9·95	10·10	4·95	5·05
35	9·70	9·85	9·75	9·90	4·85	4·95
36	9·50	9·65	9·55	9·70	4·75	4·85
37	9·25	9·40	9·30	9·45	4·65	4·75
38	9·05	9·20	9·10	9·25	4·50	4·60
39	8·80	8·95	8·85	9·00	4·40	4·50
40	8·60	8·75	8·65	8·80	4·30	4·40
41	8·40	8·55	8·40	8·55	4·20	4·30
42	8·15	8·30	8·20	8·35	4·05	4·15
43	8·05	8·25	8·00	8·20	3·95	4·05
44	7·85	8·05	7·80	8·00	3·80	3·90
45	7·60	7·80	7·55	7·75	3·70	3·80
46	7·35	7·55	7·35	7·55	3·60	3·70
47	7·10	7·30	7·15	7·35	3·50	3·60
48	6·85	7·05	6·90	7·10	3·40	3·50
49	6·65	6·85	6·70	6·90	3·30	3·40
50	6·45	6·65	6·50	6·70	3·20	3·30
51	6·25	6·45	6·25	6·45	3·10	3·20
52	6·05	6·25	6·05	6·25	3·00	3·10
53	5·85	6·05	5·85	6·05	2·90	3·00
54	5·65	5·85	5·65	5·85	2·80	2·90
55	5·45	5·65	5·45	5·65	2·70	2·80
56	5·25	5·45	5·25	5·45	2·60	2·70
57	5·05	5·25	5·05	5·25	2·50	2·60
58	4·90	5·10	4·85	5·05	2·40	2·50
59	4·70	4·90	4·70	4·90	2·30	2·40

Age at Valuation	Table I		Table II		Table III	
	H^M Mortality—Rate of Interest to be altered from					
	$3\frac{1}{2}$ % to 3 %		3 % to $2\frac{1}{2}$ %		$3\frac{1}{2}$ % to $3\frac{1}{4}$ %—$3\frac{1}{4}$ % to 3 % 3 % to $2\frac{3}{4}$ %—$2\frac{3}{4}$ % to $2\frac{1}{2}$ %	
	Increase per-cent in Values of					
	Sums Assured	Future Premiums	Sums Assured	Future Premiums	Sums Assured	Future Premiums
60	4·50	4·70	4·50	4·70	2·20	2·30
61	4·35	4·55	4·35	4·55	2·10	2·20
62	4·20	4·40	4·15	4·35	2·05	2·15
63	4·00	4·15	3·95	4·10	1·95	2·05
64	3·85	4·00	3·80	3·95	1·90	2·00
65	3·70	3·85	3·65	3·80	1·80	1·90
66	3·55	3·70	3·50	3·65	1·75	1·85
67	3·40	3·55	3·35	3·50	1·70	1·80
68	3·20	3·30	3·15	3·25	1·55	1·60
69	3·05	3·15	3·00	3·10	1·50	1·55
70	2·90	3·00	2·85	2·95	1·45	1·50
71	2·80	2·90	2·75	2·85	1·40	1·45
72	2·65	2·75	2·60	2·70	1·30	1·35
73	2·45	2·50	2·45	2·50	1·25	1·30
74	2·35	2·40	2·35	2·40	1·15	1·20
75	2·25	2·30	2·25	2·30	1·10	1·15
76	2·15	2·20	2·15	2·20	1·05	1·10
77	2·05	2·10	2·05	2·10	1·00	1·05
78	1·95	1·95	1·95	1·95	·95	·95
79	1·90	1·90	1·90	1·90	·90	·90
80	1·80	1·80	1·80	1·80	·90	·90
81	1·75	1·75	1·70	1·70	·85	·85
82	1·65	1·65	1·65	1·65	·80	·80
83	1·60	1·60	1·55	1·55	·80	·80
84	1·50	1·50	1·50	1·50	·75	·75
85	1·45	1·45	1·40	1·40	·70	·70
86	1·40	1·40	1·35	1·35	·65	·65
87	1·30	1·30	1·30	1·30	·65	·65
88	1·25	1·25	1·20	1·20	·60	·60
89	1·15	1·15	1·15	1·15	·60	·60
90	1·10	1·10	1·10	1·10	·55	·55
91	1·00	1·00	1·00	1·00	·50	·50
92	·90	·90	·90	·90	·45	·45
93	·80	·80	·80	·80	·40	·40
94	·70	·70	·70	·70	·35	·35
95	·60	·60	·60	·60	·30	·30
96	·50	·50	·50	·50	·25	·25
97	·40	·40	·40	·40	·25	·25

Note.—The use of these tables gives, approximately, the values of policies (formula, $V + \frac{P}{2}$, modifiable as shown on page 265), but *not* the values of sums assured or future premiums.

Age at Valuation	Table IV		Table V		Table VI	
	Combined H^{M} and $H^{M(5)}$ Mortality—Rate of Interest to be altered from					
	$3\frac{1}{2}$% to 3%		3% to $2\frac{1}{2}$%		$3\frac{1}{2}$% to $3\frac{1}{4}$%—$3\frac{1}{4}$% to 3% 3% to $2\frac{3}{4}$%—$2\frac{3}{4}$% to $2\frac{1}{2}$%	
	Increase per-cent in Values of					
	Sums Assured	Future Premiums	Sums Assured	Future Premiums	Sums Assured	Future Premiums
20 & under	10·40	10·40	10·50	10·50	5·25	5·25
21	10·30	10·30	10·40	10·40	5·20	5·20
22	10·20	10·20	10·30	10·30	5·15	5·15
23	10·10	10·10	10·20	10·20	5·10	5·10
24	10·00	10·00	10·10	10·10	5·05	5·05
25	9·90	9·90	10·00	10·00	5·00	5·00
26	9·80	9·80	9·90	9·90	4·95	4·95
27	9·80	9·85	9·90	9·95	4·90	4·90
28	9·80	9·90	9·90	10·00	4·85	4·85
29	9·80	9·95	9·90	10·05	4·85	4·90
30	9·80	10·00	9·90	10·10	4·85	4·95
31	9·70	9·90	9·80	10·00	4·80	4·90
32	9·60	9·80	9·65	9·85	4·70	4·80
33	9·45	9·65	9·50	9·70	4·65	4·75
34	9·30	9·50	9·35	9·55	4·55	4·65
35	9·15	9·35	9·20	9·40	4·50	4·60
36	9·00	9·20	9·05	9·25	4·40	4·50
37	8·85	9·05	8·90	9·10	4·35	4·45
38	8·65	8·85	8·75	8·95	4·25	4·35
39	8·50	8·70	8·60	8·80	4·20	4·30
40	8·35	8·55	8·45	8·65	4·10	4·20
41	8·15	8·35	8·25	8·45	4·00	4·10
42	7·95	8·15	8·00	8·20	3·90	4·00
43	7·75	7·95	7·75	7·95	3·80	3·90
44	7·55	7·75	7·50	7·70	3·70	3·80
45	7·30	7·50	7·30	7·50	3·60	3·70
46	7·10	7·30	7·10	7·30	3·50	3·60
47	6·90	7·10	6·95	7·15	3·40	3·50
48	6·70	6·90	6·75	6 95	3·30	3·40
49	6·50	6·70	6·55	6·75	3·20	3·30
50	6·30	6·50	6·35	6·55	3·10	3·20
51	6·10	6·30	6·15	6·35	3·00	3·10
52	5·95	6·15	5·95	6·15	2·90	3·00
53	5·75	5·95	5·80	6·00	2·85	2·95
54	5·60	5·80	5·60	5·80	2·75	2·85
55	5·40	5·60	5·40	5·60	2·65	2·75
56	5·20	5·40	5·20	5·40	2·60	2·70
57	5·05	5·25	5·05	5·25	2·50	2·60
58	4·85	5·05	4·85	5·05	2·40	2·50
59	4·70	4·90	4·65	4·85	2·30	2·40

Age at Valuation	Table IV		Table V		Table VI	
	Combined H^M and $H^{M(5)}$ Mortality—Rate of Interest to be altered from					
	$3\frac{1}{2}$% to 3%		3% to $2\frac{1}{2}$%		$3\frac{1}{2}$% to $3\frac{1}{4}$%—$3\frac{1}{4}$% to 3% 3% to $2\frac{3}{4}$%—$2\frac{3}{4}$% to $2\frac{1}{2}$%	
	Increase per-cent in Values of					
	Sums Assured	Future Premiums	Sums Assured	Future Premiums	Sums Assured	Future Premiums
60	4·50	4·70	4·45	4·65	2·25	2·35
61	4·35	4·55	4·30	4·50	2·15	2·25
62	4·15	4·35	4·15	4·35	2·10	2·20
63	3·95	4·10	3·95	4·10	2·00	2·10
64	3·80	3·95	3·75	3·90	1·95	2·05
65	3·65	3·80	3·60	3·75	1·85	1·95
66	3·50	3·65	3·45	3·60	1·80	1·90
67	3·35	3·50	3·30	3·45	1·70	1·80
68	3·20	3·30	3·15	3·25	1·60	1·70
69	3·05	3·15	3·00	3·10	1·50	1·60
70	2·90	3·00	2·85	2·95	1·45	1·55
71	2·75	2·85	2·75	2·85	1·40	1·50
72	2·65	2·75	2·65	2·75	1·35	1·45
73	2·50	2·60	2·55	2·65	1·25	1·35
74	2·40	2·50	2·40	2·50	1·20	1·30
75	2·30	2·40	2·25	2·35	1·15	1·25
76	2·20	2·30	2·15	2·25	1·05	1·10
77	2·10	2·20	2·05	2·15	1·00	1·00
78	1·95	1·95	1·90	1·90	·95	·95
79	1·85	1·85	1·80	1·80	·90	·90
80	1·75	1·75	1·70	1·70	·85	·85
81	1·70	1·70	1·65	1·65	·80	·80
82	1·65	1·65	1·60	1·60	·80	·80
83	1·55	1·55	1·50	1·50	·75	·75
84	1·50	1·50	1·45	1·45	·75	·75
85	1·45	1·45	1·40	1·40	·70	·70
86	1·35	1·35	1·35	1·35	·70	·70
87	1·30	1·30	1·25	1·25	·65	·65
88	1·20	1·20	1·20	1·20	·65	·65
89	1·15	1·15	1·10	1·10	·60	·60
90	1·05	1·05	1·05	1·05	·55	·55
91	·95	·95	·95	·95	·50	·50
92	·85	·85	·85	·85	·45	·45
93	·75	·75	·75	·75	·40	·40
94	·65	·65	·65	·65	·35	·35
95	·55	·55	·55	·55	·30	·30
96	·45	·45	·45	·45	·25	·25

NOTE.—The use of these tables gives, approximately, the values of policies (formula, $V+\frac{P}{2}$, modifiable as shown on page 265), but *not* the values of sums assured or future premiums.

The Effect of a Decreasing Rate of Interest on Annuities and Policy Reserves.

IN an Appendix to the Report of Messrs. A. Hendriks, A. H. Bailey, and R. P. Hardy, to the Australian Mutual Provident Society, reprinted in our columns in July 1894 (*J.I.A.*, xxi, 325), Mr. R. P. Hardy elaborated an ingenious method of valuation giving effect to a decreasing rate of interest, assumed to be subject to a constant annual reduction throughout a given period. Tables were added showing the values of annuities and assurances under the conditions mentioned, the rate of interest commencing at £3. 15*s*., and decreasing one shilling per-cent per annum until the end of the fifteenth year, after which it was assumed to remain constant at £3 per-cent per annum. A novel problem was investigated with great success in this Appendix; and, both on account of the originality with which the subject was treated, and the utility of the results, the contribution will always possess great interest for students of life assurance science.

These remarks are suggested by a perusal of a short paper read by Mr. D. P. Fackler before the Actuarial Society of America in April 1895, and reprinted in the Transactions of the Society (Vol. iv, No. 13, pp. 32–38). Mr. Fackler bases his interesting investigation upon different assumptions from those forming the foundation of Mr. Hardy's researches. He assumes "that interest for the next ten years will average $4\frac{1}{2}$ per-cent; "for the next decade 4 per-cent; and so on, dropping one half "of one per-cent each decade, until in the sixth, and thereafter "it is only 2 per-cent." Into the technical process by which the tables given in this paper were computed, we do not propose to enter, but certain of the results will be reproduced for the benefit of those who do not see the American publication.

First as to the value of life annuities, Mr. Fackler's table stands as follows:

TABLE I.—ACTUARIES' (SEVENTEEN OFFICES') MORTALITY.

Values of Life Annuities of 1 payable at the end of each year, assuming that interest will be $4\frac{1}{2}$ per-cent for the next ten years, and thereafter decrease one-half of 1 per-cent each decade, until only 2 per-cent, after which remaining constant.

Age at Issue	Interest declining	Interest 4 per-cent	Excess of 4 per-cent Values
20	18·351	18·450	·099
30	16·745	17·040	·295
40	14·706	15·093	·387
50	12·101	12·470	·369
70	9·142	9·415	·273
60	6·162	6·317	·155
80	3·596	3·661	·065

The author explains that the "old Actuaries' Table" (the Seventeen Offices' Experience, as it is called in this country) "has been taken as the basis of comparison, not because it is "believed to be a proper table for annuity calculations in general, "but because calculations on that basis could be made much "more conveniently than on any other basis, and also because it "seemed that the facts ascertained with that table as a standard "would not materially vary for a different table of mortality."

This is an inference that seems on the whole sufficiently reasonable. In regard to the main result of his comparison, Mr. Fackler proceeds, "if a company is realizing about "$4\frac{3}{4}$ per-cent on its investments at this time, after making full "allowance for the expenses necessary to the care of such invest- "ments, and it is assumed that the interest will drop to about "$4\frac{1}{4}$ by the end of the ten years, so that $4\frac{1}{2}$ per-cent will be the "average net rate realized for the next ten years, then a charge "based on a 4 per-cent table derived from annuitant experience, "with some addition for agency and general expenses, would "appear entirely safe for those ages at which annuities are "ordinarily issued."

"From this it may fairly be argued that if at any time here- "after it shall appear probable that the rate of interest will "average only 4 per-cent during the ten years following that "date, and then slowly decline thereafter, we can safely base our "annuity calculations on $3\frac{1}{2}$ per-cent."

Given this stage in the argument established, the general deduction that, under the conditions pre-supposed as to the gradual fall in the rate of interest, annuity-tariffs should be based upon a half per-cent less in the rate of interest than the probable net yield of the next ten years, is an easy one. And although Mr. Fackler does not state it in express terms, the argument will obviously apply to annuity-reserves as well as annuity-rates.

Mr. Fackler is justified in terming these results "rather surprising," more particularly in view of the vastly different effect of the same assumption as to the rate of interest upon policy-reserves, which will be demonstrated in the sequel. And in Great Britain, where considerations of the lower producing power of capital have led to the annuity-rates of our prominent companies being reviewed with some anxiety, Mr. Fackler's conclusions will prove helpful as well as instructive.

Dealing next with insurance values, Mr. Fackler produces a table of single and annual premiums based upon the same

hypothesis as to interest and mortality as before. His figures on this point may be briefly given, supplemented by the remark that the comparison with the 4 per-cent units (as in the case of annuities) fails in its object, there being no similarity or tendency to symmetry in the two series.

Table II.

Single and Annual Premiums for 1,000 of Whole-Life Insurance with interest declining from $4\frac{1}{2}$ per-cent to 2 per-cent at the rate of one-half of 1 per-cent each decade. Actuaries' (Seventeen Offices') Mortality.

Age	Single Premiums		Excess over Actuaries' 4 per-cent	Annual Premiums		Excess over Actuaries' 4 per-cent
	Interest Declining	Interest 4 per-cent		Interest Declining	Interest 4 per-cent	
20	275·62	251·91	+23·71	14·24	12·95	+1·29
30	319·08	306·17	+12·91	17·98	16·97	+1·01
40	381·26	381·04	+ 0·22	24·27	23·68	+0·59
50	470·08	481·94	−11·83	35·88	35·78	+0·10
60	579·53	599·43	−19·90	57·14	57·56	−0·42
70	696·89	718·57	−21·68	97·30	98·20	−0·90
80	802·83	820·72	−17·89	174·68	176·10	−1·42

But far more interesting than the results given in Table II is the comparison of (net-premium) reserves computed on the assumption of the declining rate of interest and the usual 4 per-cent basis, respectively. In reproducing this table (Table III) we have added a new column containing the net 3 per-cent reserves; and we venture to think that this extension of the comparison will invest the subject with some additional interest.

Table III.—Comparison of Reserves.

Based on the Assumption of a declining Rate of Interest with those derived from a 4 per-cent (and a 3 per-cent) Table. Actuaries' (Seventeen Offices') Mortality.

Age at Issue	Reserves at end of 10 Years			Reserves at end of 20 Years		
	Interest declining	4 per-cent Reserve	3 per-cent Reserve	Interest declining	4 per-cent Reserve	3 per cent. Reserve
30	125·39	107·91	126·77	296·18	253·29	285·91
40	176·99	162·97	182·25	387·18	352·84	382·67
50	237·15	226·84	245·08	482·35	456·79	481·45
60	304·51	297·42	313·11	570·58	552·49	571·06

Confining his attention to the difference between his hypothetical and the 4 per-cent reserves, Mr. Fackler directs our notice to the amount which would have to be taken out of the surplus ascertained on a 4 per-cent basis in order to provide the increased reserves on the assumed basis at the end of 10 and 20 years respectively, only ordinary whole-life contracts being taken into account. The differences are by no means insignificant. A company whose policies had been issued at an average age of 30, with a present average duration of anything between 10 and 20 years, would have to draw upon its surplus (under whole-life contracts) to the extent of one-sixth of its total estimated liability to enable it to change the basis of its reserves from 4 per-cent to the declining rate of interest. Say its (whole-life) funds are £1,750,000 and its 4 per-cent liability £1,500,000, the whole of the surplus would disappear in the adjustment. These figures are, of course, purely imaginary and illustrative, but they may help us to appreciate the position.

Clearer still is the lesson to be drawn from the comparison when the 3 per-cent reserves are placed in juxtaposition with the hypothetical. At ages 30 and 40 at issue, the reserves at the end of ten years are greater by about 2 per-cent in the case of the 3 per-cent basis, and at the end of 20 years are greater by about 2 per-cent in the case of the hypothetical basis. If 35 be assumed as a reasonable average age at issue, and 15 years as an average duration, it may be inferred that the hypothetical and the 3 per-cent reserves are, roughly speaking, equivalent.

This is a conclusion of very great importance. Its application is as pertinent on this side of the Atlantic Ocean as on the other. It shows that if a fall in the average rate of interest from $4\frac{3}{4}$ at the present moment to 2 per-cent eventually, by one-half per-cent each decade, is a shrewd and well-founded forecast of the future yield of capital in America, companies who wish to make due provision for the impending change should calculate their reserves upon a 3 per-cent basis. To us in the United Kingdom, with a lower rate ruling at the present time, and with perhaps no greater probability of maintaining that rate, the result just deduced is a forcible argument in favour of strong reserves. By learning how little influence the present rate of interest exercises on the reserves in the face of a slow but steady decline, we may be induced to pay a less exclusive regard to the circumstances of the moment and to reckon betimes with this great contingency of the future.

G. H. R.

*The Employment of Interpolation in Statistics.** By Harald Westergaard.

[Translated and abridged by D. A. Bumsted, F.I.A.]

[After some interesting preliminary remarks on the subject of statistics generally, and the utility of interpolation in their preparation, Professor Westergaard points out that in vital statistics the enumeration of the population at each year of age is seldom wanted, and, in fact, owing to the well known exaggeration in census returns of the numbers living at decennial ages, an adjustment is generally needed. He then proceeds:] The decennial reports by the English Registrar-General of births, deaths and marriages, contain full materials for the study of the mortality of professional men. The report published in 1875 gives a very detailed classification of ages, but it is not so with the next decennial report, which divides the ages into five periods only—15–20, 20–25, 25–45, 45–65, 65 and upwards.

In the following remarks we shall, for the sake of brevity, describe the latter as 20-year periods, as distinguished from the divisions 25–35, 35–45, &c., 10-year periods. As a rule the 20-year periods are not short enough to give satisfactory results. Take for example a class like the farmers (Farmers & Graziers—Report of 1875, p. 175, Table 64).

The following Table has been prepared from the figures there given—

Table I.

Ages	Years Lived	Deaths	Annual Deaths out of 10,000 Persons living			Expected Deaths among Farmers using	
			General Population		Farmers	Periods of 10 Years	Periods of 20 Years
15–25	16,597	181	74		109	123	123
25–35	86,519	765	98	114	88	848	2,571
35–45	139,016	1,252	130		90	1,807	
45–55	156,787	1,964	185	240	125	2,901	7,173
55–65	142,089	3,324	322		234	4,575	
65–75	94,921	5,420	668	944	571	6,341	13,072
75 and above	43,554	7,386	1,658		1,696	7,221	
Total . .	679,483	20,292	...		...	23,816	22,939

* *Die Anwendung der Interpolation in der Statistik.*

Through the large preponderance of lives at the older ages which characterizes the life-curve of farmers, the expected mortality is postponed if we use 20-year periods instead of those for 10 years. The mortality co-efficient* of farmers is, owing to this distribution of lives greater in 20-year periods than it would be if the life-curve were the same as in the general population. Their expected mortality will consequently be less in 20-year periods than in 10-year. The total difference is 4 per-cent, for although the expected mortality calculated on the basis of 10-year periods exceeds the actual by 17 per-cent, that by 20-year periods exceeds the actual by only 13 per-cent. It thus appears that by using the larger age periods fallacious conclusions may easily be drawn as to the mortality among various occupations, even though the deviation from the average be less than in the above instance.

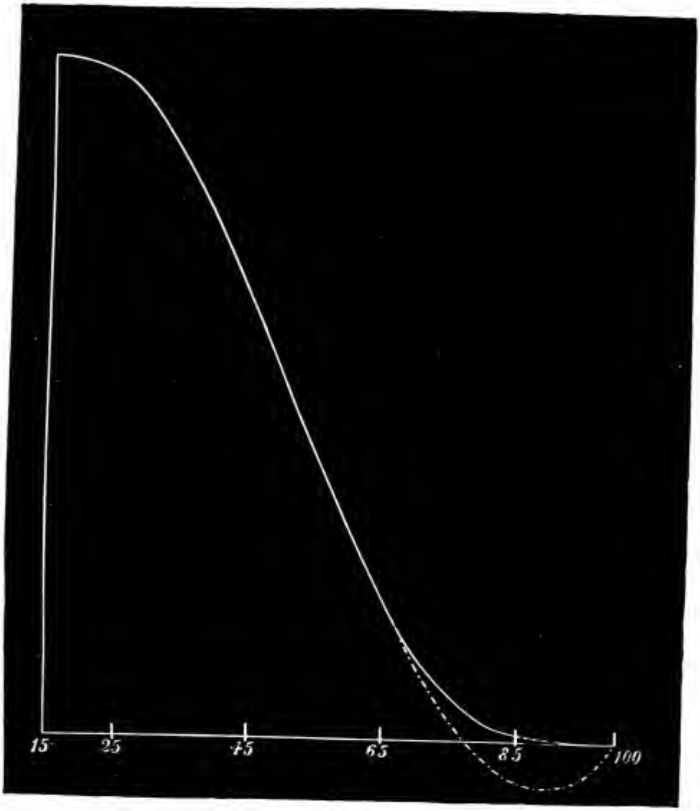
If, therefore, the observations are only given for periods of 20 years (as in the Report for 1885), we must calculate by interpolation the probable effect of division into smaller (say 10-year) groups of ages.

Let y_x be the number of persons aged x and upwards in a given class of the population. This quantity clearly corresponds with the numbers living at age x and upwards in a mortality table. If then we know the value of y_x for certain values of x, we can calculate by means of Newton's formula† (or any other suitable formula of interpolation) the corresponding values of y_x for any other values of x, assuming y_x to be an integral algebraic function of x.

This method is sufficiently exact when short periods of life are dealt with. If the values over a lengthened period of life are required, however, the assumption generally fails. In order to give a clear idea of this, I have represented the age-periods in the Farmers' Table graphically in the following diagram, in which the values of y_x are represented as ordinates, and the values of x as abscissæ.

[* The function designated by Professor Westergaard as "Mortality Co-efficient" appears to correspond with our "Central Death Rate."—Ed. *J.I.A.*]

† [*J.I.A.*, xiv, 28.]



With the exception of the irregularity at ages 15-25, this is a typical form of a life-curve; a rapid fall from age 25 to about 70; then a slower decline during which the curve gradually approaches the axis of x.

The following points in the curve being known:

$$y_{15} = 10{,}000 \qquad y_{45} = 6{,}437$$
$$y_{25} = 9{,}756 \qquad y_{65} = 2{,}038$$

we have to find other points in it. Newton's formula yields very satisfactory results up to age 65: for example, at age 35 we find $y = 8{,}426$,* the actual value being 8,483. At age 55 we find 4,185* instead of 4,130. For the higher ages the results are not satisfactory. At age 75 we have 336,* the actual number being

* There appears to be some miscalculation in the numbers here given. By Newton's formula (to third differences) the numbers are respectively $y_{35} = 8{,}434$, $y_{55} = 4{,}170$, $y_{75} = 444$.—D. A. B.

641; and soon afterwards negative values are obtained, which are perfectly useless. These are indicated by the dotted line.

[The author then states that this example shews that Newton's formula does not in all cases give sufficiently close results, and suggests some modifications, with the view of making it more suitable. The graphic method, although most useful for preliminary investigations, is stated to be not sufficient as a rule for final results. The author proceeds:]

My proposal consists, firstly, in resolving the function y_x into two factors. Assume $y_x = u_x z_x$, where u_x is a function corresponding in some degree with the typical curve, z_x is an integral algebraic function to be calculated by Newton's formula, or one which can easily be reduced into this form. We may assume a value of u_x so that z_x becomes nearly equal to 1, and for this purpose either calculate the constants of u_x by means of the method of least squares, or choose certain points for which we have exactly $z_x = 1$.

That this may be practicable we must use such forms of the function u_x as will facilitate the calculation; otherwise it may be very laborious. Either we must lighten the work by calculating once for all a convenient supplementary table similar to a table of logarithms, or the formula must be very simple. But I think there is no probability that any practically useful analytic functions will be found.

We need not, however, find the mathematical form of the function u_x. It is sufficient to know empirical values of it—and this is the second point of my proposal.* Such empirical table can be derived from a population having a characteristic life-curve, the latter having a typical form at the higher ages. It is, of course, important to make a careful selection, and several trials will be required before a satisfactory form is found.

Having determined empirical values of u_x, we obtain from $y_x = z_x . u_x$ where $z_x = 1 + ax + bx^2 + \&c. = 1 + xa_x$, the following equation:

$$\frac{y_x - u_x}{xu_x} = a_x = \frac{z_x - 1}{x}. \quad \dagger$$

[* It would be interesting to compare Professor Westergaard's suggested method with that adopted by Mr. G. F. Hardy, and mentioned in his paper on "The Rates of Mortality among the Natives of India" (*J.I.A.*, XXV, 229).—Ed. *J.I.A.*]

[† It is not apparent that any advantage is gained by the use of this function (a_x). The direct interpolation of z_x (or $z_x - 1$) would be the more obvious course. In Table III a column giving the figures by this method is added, from which it appears that the results are somewhat more satisfactory.—Ed. *J.I.A.*]

For u_x we have the table of observations: y_x is the required function of which only certain values are found. These furnish the necessary equations for the calculation of the constants a, b, &c.

Let us now take as an example, the life-curve of the general male population, as expressing the empirical function u_x making $x=0$ for age 15, $x=1$ for age 25, &c. We know that of 10,000 persons

9,756	are aged	25	and upwards	($x=1$)
6,437	,, ,,	45	,, ,,	($x=3$)
2,038	,, ,,	65	,, ,,	($x=5$)

and we then have the following table:

TABLE II.

Age		Number of Males aged w and upwards out of 10,000 Males aged 15 and upwards		$y_x - u_x$	$x \, . \, u_x$	a_x
x	w	Farmers y_x	General Population u_x			
1	(25)	9,756	7,065	2,691	7,065	·3809
2	(35)	...	4,798	...	9,596	...
3	(45)	6,437	2,970	3,467	8,910	·3891
4	(55)	...	1,597	...	6,388	...
5	(65)	2,038	681	1,357	3,405	·3985
6	(75)	...	193	...	1,158	...

From these values we obtain by Newton's formula the following values of y_x:

Age	Interpolated Numbers	Actual Numbers
35	8,491	8,483
55	4,112	4,130
75	660	641

giving the distribution of the 10,000 lives thus:

TABLE III.

Age	Numbers according to			Percentage of Differences
	Observations	Interpolation	By direct Interpolation of z_x	
15–25	244	(244)	(244)	...
25–35	1,273	1,265	1,267	− ·6
35–45	2,046	2,054	2,052	+ ·4
45–55	2,307	2,325	2,325	+ ·8
55–65	2,092	2,074	2,074	− ·8
65–75	1,397	1,378	1,378	−1·4
75–	641	660	660	+3·0
Total .	10,000	10,000	10,000	...

To some extent the suggested process has been used before. Political arithmeticians have frequently applied to other districts the figures derived from one district, often indeed with little justification. Their calculations are usually simple, but complicated problems occasionally arise. For example, in his "Göttliche Ordnung", Vol. II, pp. 314 *et seq.*, Süssmilch calculated a table of the distribution of deaths at each age "proportioning" the numbers according to another more detailed table. Thus he distributes 59 deaths at ages 2–5 years among the single ages, assuming the same distribution of deaths as in a second Table, which shows for those ages 46 deaths, or 29 and 17. He then easily finds 29, 19 and 11 deaths, but the next group does not correspond with this distribution, causing an upward bound which Süssmilch equalizes by a treatment which is undoubtedly very rough.

For instance, at age six the number of deaths according to the "proportioning" process is 14, but is reduced by Süssmilch to 10. The defect in his method is that it does not take into account (as mine does) all the groups of ages together, but works out a separate calculation for each group. Besides, the method of interpolation which I propose would be inapplicable in this instance, because the life-curves of the two tables are very different. This shows the necessity of proceeding with great caution in the selection of the function u_x.

The method analyzed above can be best applied to a table of the numbers living, but not so easily to a table of the deaths. The reason of this is, firstly, that the distribution of the deaths is usually devoid of typical form, which the numbers living have: and, secondly, that accidental variations may exercise an influence. Still, we can often obtain very good results if we leave out of account the years of infancy, as is done in the English occupation statistics. For instance, we use the mortality intensity of the general population to calculate the deaths expected in the occupation concerned.

TABLE IV.

Ages	Actual No. of Years Lived	Interpolated No. of Years Lived	Expected No. of Deaths on basis of	
			Actual Years Lived	Interpolated Years Lived
15–25	16,597	(16,597)	123	(123)
25–35	86,519	85,955	848	842
35–45	139,016	139,566	1,807	1,814
45–55	156,787	157,980	2,901	2,923
55–65	142,089	140,925	4,575	4,538
65–75	94,921	93,633	6,341	6,255
75–100	43,554	44,846	7,221	7,435
Total . .	679,483	679,502	23,816	23,930

The total difference is, therefore, under ½ per-cent, and is quite insignificant in comparison with the difference found on the basis of intervals of 20 years.

As evidence of the applicability of the method to the death cases, I will give the result of a calculation respecting the Farmers. The whole number of deaths was 20,292, distributed as follows, according to observation and calculation.

TABLE V.

Ages	Distribution of Deaths among Farmers according to	
	Observations	Calculation
	181	(181)
25– 35	765	741
35– 45	1.252	1,276
45– 55	1,964	1,942
55– 65	3,324	3,346
65– 75	5,420	5,471
75–100	7,386	7,335
Total .	20,292	20,292

Here also the exactitude obtained is clearly quite satisfactory. The mortality coefficients found on the basis of these figures, and of the observed and interpolated population numbers, are as follows:

TABLE VI.

Ages	Out of 10,000 Persons at all ages, there died yearly according to	
	Observations	Interpolation
15–25	109	(109)
25–35	88	86
35–45	90	91
45–55	125	123
55–65	234	237
65–75	571	585
75	1,696	1,633

All who are familiar with mortality investigations will admit that the results thus obtained by interpolation are sufficiently exact.

Innkeepers are a class whose life-curve differs widely from the general population. Among the latter 29 per-cent are aged 15–25; among innkeepers only 5 per-cent. Many persons do not engage in this occupation until mature age; consequently the numbers at ages 35–45 show a large increase, which afterwards diminishes, since among innkeepers there are proportionately fewer persons over 75 than in the general population.

The distribution of deaths in that occupation also varies remarkably from the average. We should therefore advise that another population group be used as a basis for u_x. It will not be uninteresting to make a comparison with the general population. In order to avoid wearying the reader I will only set forth the results. The following table shows how many deaths among 100,000 persons of the general population are expected according to the various tables of mortality:

TABLE VII.

Ages	Expected Deaths on basis of		
	Actual Mortality		Interpolated Table
	20-Year Periods	10-Year Periods	10-Year Periods
15– 25	(282)	(282)	(282)
25– 45	737	702	716
45– 65	781	787	782
65–100	714	785	794
Total	2,514	2,556	2,574

The deviations in the last two columns are only 1 or 2 per-cent, while those of the 1st and 2nd rise to 9. The total difference between the 2nd and 3rd columns amounts to only 7 per-cent, but between the 1st and 2nd it is 1·6 per-cent. Interpolation yields, therefore, very good results in comparison with those obtained from the table with 20-year intervals.

It will be found that in isolated cases the difference in the calculation on the basis of 20-year periods will be so small that no advantage can be gained by interpolation.

Usually, however, interpolation yields better results than the actual table with 20-year age periods, even if the function u_x has been chosen without special consideration.

The results of some calculations derived from the Danish population statistics will furnish another aspect of the proposed interpolation. I have selected three classes which differ materially in their life-curves from that of the general population, namely, the populations of Copenhagen, the Danish West Indies, and Greenland. I assume that we have the numbers living of the general population (males) for the year 1890 in groups of 5 years, but for the special populations, only 6 groups of ages, namely 0–5 (that of childhood), 5–25 (in which travelling plays a great part), 25–45, 45–65, 65–85 and over 85. In the following table the results of the calculation are shown side by side with the actual numbers.

Table VIII.

Ages	Copenhagen (Males)		Greenland (both sexes)		West Indies (both sexes)	
	Observations	Interpolation	Observations	Interpolation	Observations	Interpolation
0– 5	1,299	(1,299)	1,355	(1,355)	910	(910)
5– 15	2,052	1,998	2,422	2,519	1,932	1,923
15– 25	1,748	1,802	2,110	2,013	1,856	1,865
25– 35	1,754	1,718	1,544	1,616	1,841	1,776
35– 45	1,354	1,390	1,291	1,219	1,407	1,472
45– 55	880	901	785	784	978	1,003
55– 65	544	523	377	378	585	560
65– 75	294	281	109	110	344	333
75– 85	68	81	7	6	120	131
85– 90	6	6	0	0	20	22
90–100	1	1	0	0	7	5
Total	10,000	10,000	10,000	10,000	10,000	10,000

The average age, according to the actual numbers, varies by one month for Copenhagen from that calculated by the interpolated numbers. For Greenland, the variation is two months—for the West Indies, three weeks. Compared with the other sources of error which occur, especially in remote places, such as Greenland and the West Indies, these deviations are practically insignificant.

As a last example, I select a population which exhibits a decidedly different arrangement of lives from the general population, namely, married persons. For groups of ages I choose 15–25, 25–45, 45–65, 65–85, and 85 and upwards. On the basis of the observations distributed according to these periods we obtain the following numbers:

TABLE IX.

Ages	Distribution of 10,000 Males above 15 Years of Age		
	The Whole Population	Married Men	
		According to Observations	According to Calculation
15– 25	2,635	187	(187)
25– 35	2,094	2,233	2,255
35– 45	1,718	2,656	2,634
45– 55	1,437	2,225	2,249
55– 65	1,078	1,574	1,550
65– 75	743	911	896
75– 85	235	199	214
85–100	30	15	15
Total .	10,000	10,000	10,000

The average age of the married population, as found by observation, differs from that calculated by the interpolated numbers by only 1½ weeks.

It would be very interesting to apply the proposed method in other directions. For example, the distribution of the population according to income, of buildings according to sums insured, &c. The difficulty here consists in the fact that usually an excessive accumulation is found in the lowest classes, and it would probably be necessary to employ very restricted divisions. The task has, however, few attractions, as it has little practical application. I forbear, therefore, to pursue it further, but I may assert generally

that, other circumstances being equal (with the same intervals, &c.), the proposed method will, as a rule, yield better results than ordinary interpolation. It is possible that an exhaustive treatment of the whole subject would disclose further practical methods of interpolation.

CORRESPONDENCE.

ON SOME PRACTICAL APPLICATIONS OF SIMPLE INTERPOLATION FORMULÆ.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The recent publication, by the Institute of Actuaries, of Joint-life Annuity values according to the experience of the Government Life Annuitants, tabulated, as regards one of the two lives involved, for quinquennial intervals of age only, has directed my attention to some simple practical applications of well-known interpolation formulæ, which may be of interest to Students of the Institute, and generally to those employing the tables in question.

Let there be given the following successive values, at quinquennial intervals, of any tabular function of two independent variables x and y, so that

$$\phi_{x.y}=a, \qquad \phi_{x+5.y}=c,$$

$$\phi_{x.y+5}=b, \qquad \phi_{x+5.y+5}=d,$$

and let it be required to deduce an approximate value for $\phi_{x+m.y+n}$, where m and n are each less than 5.

We have, interpolating to first differences only,

$$\phi_{x.y+n}=\phi_{x.y}+\frac{n}{5}(\phi_{x.y+5}-\phi_{xy})$$

$$=\tfrac{1}{5}\{(5-n)a+nb\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, $$\phi_{x+5.y+n}=\tfrac{1}{5}\{(5-n)c+nd\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence we may deduce

$$\phi_{x+m.y+n}=\phi_{x.y+n}+\frac{m}{5}(\phi_{x+5.y+n}-\phi_{x.y+n})$$

$$=\tfrac{1}{5}\{(5-m)\phi_{x.y+n}+m\phi_{x+5.y+n}\} \quad . \quad . \quad . \quad (3)$$

and inserting the corresponding values from formulæ (1) and (2), and reducing, we have finally

$$\phi_{x+m.y+n}=\tfrac{1}{25}\{(5-m)(5-n)a+n(5-m)b+m(5-n)c+mnd \quad . \quad (4)$$

an expression which gives the approximate value of the function sought in terms of the *values* of the four given tabular functions instead of their *differences*, and which will sometimes be found in practice more convenient than the usual formulæ involving the differences of the functions.

The following table gives the values of the multipliers of a, b, c, and d respectively, for all values of m and n from 0 to 4 inclusive:

Given $\phi_{x.y}=a$, $\phi_{x.y+5}=b$, $\phi_{x+5.y}=c$, $\phi_{x+5.y+5}=d$.

To find $\phi_{x+m.y+n}$ (m and n each <5).

Values of n	Values of m																			
	0				1				2				3				4			
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d
0	5	0	0	0	4	0	1	0	3	0	2	0	2	0	3	0	1	0	4	0
1	4	1	0	0	16	4	4	1	12	3	8	2	8	2	12	3	4	1	16	4
2	3	2	0	0	12	8	3	2	9	6	6	4	6	4	9	6	3	2	12	8
3	2	3	0	0	8	12	2	3	6	9	4	6	4	6	6	9	2	3	8	12
4	1	4	0	0	4	16	1	4	3	12	2	8	2	8	3	12	1	4	4	16

NOTE.—The divisor, to obtain the interpolated value, is equal to the sum of the four multipliers of a, b, c, and d, which is always either $5\left(=\frac{1}{\cdot 2}\right)$ or to $25\left(=\frac{1}{\cdot 04}\right)$.

This table is useful for reference, and in deducing interpolated values of any tabular function of two variables, where the given values, for both or for either of the variables, are tabulated at quinquennial intervals only. For example, the office rates for two-life assurances and for two-life annuities (whether joint, last survivor, or contingent) are frequently tabulated for every fifth age only of each of the two lives involved, and the ordinary process of deducing the interpolated values at intermediate ages, by taking proportionate parts of the differences of the given values of the function, is somewhat cumbersome, and requires care to obtain a correct result.

An example or two will make clear the practical application of the values given in the table:

(1). Given

$$P_{30.40}=4\cdot433=a,$$
$$P_{30.45}=5\cdot049=b,$$
$$P_{35.40}=4\cdot688=c,$$
$$P_{35.45}=5\cdot265=d.$$

To find $P_{33.42}$. Here $m=3$, $n=2$, and we have, approximately,

$$P_{33.42}=\tfrac{1}{25}(6a+4b+9c+6d)$$

$$\begin{aligned} 6\times 4\cdot433 &= 26\cdot598 \\ 4\times 5\cdot049 &= 20\cdot196 \\ 9\times 4\cdot688 &= 42\cdot192 \\ 6\times 5\cdot265 &= 31\cdot590 \\ \hline & \quad 120\cdot576\times\cdot 04=4\cdot823 \end{aligned}$$

the true value being 4·784.

By the ordinary method of differencing, and taking proportional parts, the process would be as follows:

$$\begin{array}{rlrl} P_{30.40}= & 4{\cdot}433 & P_{35.40}= & 4{\cdot}688 \\ P_{30.45}= & 5{\cdot}049 & P_{35.45}= & 5{\cdot}265 \\ \hline \Delta= & +{\cdot}616 & \Delta= & +{\cdot}577 \\ {\cdot}4\Delta= & +{\cdot}246 & {\cdot}4\Delta= & {\cdot}231 \\ \hline P_{30.42}= & 4{\cdot}679 & P_{35.42}= & 4{\cdot}919 \end{array}$$

$$\begin{array}{rl} P_{30.42}= & 4{\cdot}679 \\ P_{35.42}= & 4{\cdot}919 \\ \hline \Delta= & +{\cdot}240 \\ {\cdot}6\Delta= & +{\cdot}144 \\ \hline P_{33.42}= & 4{\cdot}823 \text{ approximately.} \end{array}$$

(2). Required the approximate value, at date of purchase, of an annuity on the lives of a male aged 43 and a female aged 56, according to the Government Life Annuity Experience, with interest at 3 per cent.

Here we have, from the recently published Institute Tables:

$$a_{[56:41]}=10{\cdot}948=a$$

$$a_{[56:46]}=10{\cdot}596=b$$

also $m=0$, $n=2$, whence, approximately,

$$a_{[56:43]}=\tfrac{1}{5}(3a+2b)$$

$$\begin{array}{r} 3\times 10{\cdot}948=32{\cdot}844 \\ 2\times 10{\cdot}596=21{\cdot}192 \\ \hline 54{\cdot}036\times{\cdot}2=10{\cdot}807 \end{array}$$

By the usual method of differences, the process would be

$$\begin{array}{rl} a_{[56:41]}= & 10{\cdot}948 \\ a_{[56:46]}= & 10{\cdot}596 \\ \hline \Delta= & -{\cdot}352 \\ {\cdot}4\Delta= & -{\cdot}141 \\ \hline a_{[56:43]}= & 10{\cdot}807 \text{ approximately.} \end{array}$$

There is here little to choose between the two methods, as regards labour, but the former process involves less liability to error from the use of signs, or the calculation of the proportionate parts.

The approximate values arrived at by interpolation to first differences only are, of course, not very accurate. If *second* differences are introduced, the resulting formulæ are, in cases where

m and n have both a significant value, somewhat cumbersome, as involving five tabular values of the function, and high multipliers. In the case, however (as in the recently published Government Annuity Tables) where $m=0$, that is, where the values for the one life are tabulated for every age, and for the other life at quinquennial intervals of age only, the formulæ and multipliers to second differences are more simple, and may be sometimes usefully employed where greater accuracy is required in the interpolated value.

For this purpose, alternative formulæ may be employed, the one, proceeding by ordinary differences:

$$\phi_{x.y+n}=\phi_{xy}+\frac{n}{5}(\phi_{x.y+5}-\phi_{xy})-\left(\frac{n}{10}-\frac{n^2}{50}\right)(\phi_{x.y+10}-2\phi_{x.y+5}+\phi_{xy})$$

$$=\tfrac{1}{50}\{(50-15n+n^2)\phi_{xy}+(20n-2n^2)\phi_{x.y+5}-(5n-n^2)\phi_{x.y+10}\}. \quad (5)$$

the other, proceeding by central differences, as developed by Mr. Woolhouse, in his classical paper (*J.I.A.*, xi, 61, 301) upon the subject of interpolation:

$$\phi_{x.y+n}=\phi_{xy}+\frac{n}{5}(\phi_{x.y+5}-\phi_{xy})-\left(\frac{n}{10}-\frac{n^2}{50}\right)(\phi_{x.y-5}-2\phi_{xy}+\phi_{x.y-5})$$

$$=\tfrac{1}{50}\{(50-2n^2)\phi_{xy}+(5n+n^2)\phi_{x.y+5}-(5n-n^2)\phi_{x.y-5}\} \quad . \quad . \quad (6)$$

It will be observed that by the formula (5) to ordinary second differences, the tabular values employed for deducing the interpolated value $\phi_{x.y+n}$ are ϕ_{xy}, $\phi_{x.y+5}$, and $\phi_{x.y+10}$; while by the formula (6) to central second differences, the values employed are $\phi_{x.y-5}$, ϕ_{xy}, and $\phi_{x.y+5}$.

For successive values of n from 0 to 4 inclusive, the multipliers by formulæ (5) and (6) respectively, are shown in the following tables:

Ordinary Differences.

Val of n	$\phi_{x.y}$	$\phi_{x.y-5}$	$\phi_{x.y-10}$
0	1	0	0
1	18	9	−2
2	12	16	−3
3	7	21	−3
4	3	24	−2

Central Differences.

Values of n	$\phi_{x.y-5}$	$\phi_{x.y}$	$\phi_{x.y-5}$
0	0	1	0
1	−2	24	3
2	−3	21	7
3	−3	16	12
4	−2	9	18

In order to illustrate the relative closeness of the approximation, the several formulæ numbered (1), (5), and (6), may now be applied to the computation of the value of a joint-life annuity according to the H^M 3 per-cent tables. Let it be required for example, to deduce approximate values of $a_{50.48}$, given the values of $a_{50.40}$, $a_{50.45}$, $a_{50.50}$ and $a_{50.55}$.

By formula (1) to first differences,

$$a_{50.48}=\tfrac{1}{5}(2a_{50.45}+3a_{50.50})$$
$$=10{\cdot}8331.$$

By formula (5) to second differences,

$$a_{50.48}=\tfrac{1}{25}(7a_{50.45}+21a_{50.50}-3a_{50.55})$$
$$=10{\cdot}8578.$$

By formula (6) employing central second differences,

$$a_{50.48}=\tfrac{1}{25}(16a_{50.45}+12a_{50.50}-3a_{50.40})$$
$$=10{\cdot}8543.$$

The true value, as given in the Institute H^M 3 per-cent tables, is 10·8535; and it will be seen that, as stated by Mr. Woolhouse (*J.I.A.*, xi, 73 [note]), the central difference formula gives a more accurate result than that derived from the employment of ordinary second differences.

I am, Sir,
Yours, &c.,
THOMAS G. ACKLAND.

Addiscombe, Croydon,
30 *December* 1895.

GRADUATION—MR. J. A. HIGHAM'S THEOREM.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In the study of graduation formulas, I have been rather surprised that comparatively little use has been made of the very remarkable theorem given by Mr. J. A. Higham (*J.I.A.*, xxv, 17 and 245–8), by which the result of any number of successive summations of the terms of a series is expressed by means of the first term of the series and its differences. Mr. Higham himself has fully explained the application of his theorem for purposes of graduation, and in his last contribution on the subject (*J.I.A.*, xxxi, 319) he has shown how it may be made to yield Woolhouse's formula. The demonstration of the theorem which Mr. Higham gives seems to me unnecessarily difficult, and I therefore venture to submit the following simpler proof.

Suppose the terms of the series $u_0, u_1, u_2, \ldots u_n$ to be summed continuously in p's; the results to be summed in q's; the next results to be summed in r's, and so on, till the series has been reduced to

one term only: it is required to find a general expression for the value of this term, which we shall denote by S.

The terms of the original series may be written

$$u_0,\ (1+\Delta)u_0,\ (1+\Delta)^2u_0,\ \ldots\ (1+\Delta)^n u_0,$$

and therefore the sum of the first p terms is

$$\{1+(1+\Delta)+(1+\Delta)^2\ \ldots\ +(1+\Delta)^{p-1}\}u_0,$$

or $$\frac{(1+\Delta)^p-1}{\Delta}u_0, \quad \text{which may be denoted by } \mathrm{S}_p.$$

S_p, then, is the first term of the new series got by the summation of the original series in p's, and it is clear that the second term of this new series is simply $(1+\Delta)\mathrm{S}_p$, and that the others are $(1+\Delta)^2\mathrm{S}_p$, $(1+\Delta)^3\mathrm{S}_p$, ... $(1+\Delta)^{n-p+1}\mathrm{S}_p$.

Similarly, if the terms of this new series are summed in q's, a third series is got, the first term of which is S_{pq}, where

$$\mathrm{S}_{pq}=\frac{(1+\Delta)^q-1}{\Delta}\mathrm{S}_p=\frac{(1+\Delta)^p-1}{\Delta}\cdot\frac{(1+\Delta)^q-1}{\Delta}\cdot u_0.$$

and the remaining terms are

$$(1+\Delta)\mathrm{S}_{pq},\ (1+\Delta)^2\mathrm{S}_{pq},\ \ldots\ (1+\Delta)^{n-p-q+2}\mathrm{S}_{pq}.$$

Proceeding thus, it is clear that finally we get

$$\mathrm{S}=\frac{(1+\Delta)^p-1}{\Delta}\cdot\frac{(1+\Delta)^q-1}{\Delta}\cdot\frac{(1+\Delta)^r-1}{\Delta}\ \ldots\ u_0,$$

where, if t summations in all have been made, we must have

$$p+q+r+\ \ldots\ =n+t.$$

Let now $p^2+q^2+r^2+\ldots$ be denoted by s_2, and let us denote by σ_m the sum of the mth powers of $(p-1)$, $(q-1)$, so that

$$\begin{aligned}\sigma_1&=(p-1)+(q-1)+\ \ldots\\ &=p+q+r\ \ldots\ -t=n\\ \sigma_2&=(p-1)^2+(q-1)^2\ \ldots\\ &=s_2-2(n+t)+t\\ &=s_2-2n-t.\end{aligned}$$

Then

$$\begin{aligned}\frac{\mathrm{S}}{pqr\ldots}=&\left\{1+\frac{p-1}{2}\Delta+\frac{p-1.p-2}{6}\Delta^2+\frac{p-1.p-2.p-3}{24}\Delta^3+\ \ldots\right\}\\ &\times\left\{1+\frac{q-1}{2}\Delta+\frac{q-1.q-2}{6}\Delta^2+\frac{q-1.q-2.q-3}{24}\Delta^3+\ \ldots\right\}\\ &\times\left\{1+\frac{r-1}{2}\Delta+\frac{r-1.r-2}{6}\Delta^2+\frac{r-1.r-2.r-3}{24}\Delta^3+\ \ldots\right\}\\ &\times\left\{\ \ldots\ldots\ldots\ldots\ldots\ldots\ \right\}u_0.\end{aligned}$$

In the expansion of the right-hand side of this equation,

the coefficient of Δ is $\frac{1}{2}\sigma_1 = \frac{n}{2}$;

$$\text{the coefficient of } \Delta^2 \text{ is } \tfrac{1}{4}\Sigma(p-1)(q-1) + \tfrac{1}{6}\Sigma(p-1)(p-2)$$
$$= \tfrac{1}{4}\cdot\tfrac{1}{2}(\sigma_1^2-\sigma_2) + \tfrac{1}{6}\{\Sigma(p-1)^2 - \Sigma(p-1)\}$$
$$= \tfrac{1}{8}(n^2 - s_2 + 2n + t) + \tfrac{1}{6}(s_2 - 2n - t - n)$$
$$= \frac{n(n-2)}{8} + \frac{s_2-t}{24};$$

the coefficient of Δ^3 is

$$\Sigma\frac{p-1.q-1.r-1}{8} + \Sigma\frac{p-1.p-2.q-1}{12} + \Sigma\frac{p-1.p-2.p-3}{24},$$

of which the first term is

$$\tfrac{1}{8}\cdot\tfrac{1}{3}\{\Sigma(p-1).\Sigma(p-1)(q-1) - \Sigma(p-1)^2.(q-1)\}$$
$$= \tfrac{1}{24}\{\Sigma(p-1).\Sigma(p-1)(q-1) - \Sigma(p-1)^2.\Sigma(p-1) + \Sigma(p-1)^3\}$$
$$= \tfrac{1}{24}\left\{\frac{\sigma_1(\sigma_1^2-\sigma_2)}{2} - \sigma_2\sigma_1 + \sigma_3\right\};$$

the second term is $\tfrac{1}{12}\Sigma(p-1)^2.(q-1) - \tfrac{1}{6}\Sigma(p-1)(q-1)$

$$= \tfrac{1}{12}(\sigma_1\sigma_2 - \sigma_3) - \tfrac{1}{12}(\sigma_1^2 - \sigma_2),$$

and the third term is $\tfrac{1}{24}\Sigma(p-1)(p-1-1)(p-1-2)$

$$= \tfrac{1}{24}(\sigma_3 - 3\sigma_2 + 2\sigma_1).$$

Hence the coefficient of Δ^3 is $\dfrac{\sigma_1^3}{48} - \dfrac{\sigma_1^2}{12} + \dfrac{\sigma_1}{12} + \dfrac{\sigma_1\sigma_2}{48} - \dfrac{\sigma_2}{24}$

$$= \frac{\sigma_1(\sigma_1-2)^2}{48} + \frac{\sigma_2(\sigma_1-2)}{48}$$
$$= \frac{(\sigma_1-2)}{48}\{\sigma_1(\sigma_1-4) + \sigma_2 + 2\sigma_1\}$$
$$= \frac{n(n-2)(n-4)}{48} + \frac{(n-2)(s_2-t)}{48}.$$

If now we assume that the original series proceeds by *third* differences only, so that $\Delta^4 u_0 = \Delta^5 u_0 = \ldots = 0$, we get

$$\frac{8}{pqr\ldots} = \left[1 + \frac{n}{2}\Delta + \left\{\frac{n(n-2)}{8} + \frac{s_2-t}{24}\right\}\Delta^2 + \left\{\frac{n(n-2)(n-4)}{48} + \frac{(n-2)(s_2-t)}{48}\right\}\Delta^3\right] u_0,$$

which is Mr. Higham's expression with a slightly modified notation.

I am, Sir,
Yours faithfully,
ABRAHAM LEVINE.

National Life Assurance Society,
5 *December* 1895.

UNIFORM SENIORITY.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—In Part II of the *Text-Book*, Mr. King gives an investigation into the most general law of human mortality which will give Simpson's rule for joint-life annuities, and deduces as that law the function of Gompertz. It has occurred to me that a similar investigation into the most general law of mortality that will permit the substitution of *two lives of equal ages* for any two given lives, might perhaps interest some of the readers of the *Journal*.

In order to make this substitution we must be able, for any given values of x and y, to determine w, so that ${}_np_{ww}={}_np_{xy}$ for all values of n. This may be otherwise expressed thus:

$$2\log{}_np_w=\log{}_np_x+\log{}_np_y$$

for all values of n. Differentiating with respect to n, and changing the sign of both sides of the equation, we get

$$2\mu_{w+n}=\mu_{x+n}+\mu_{y+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for all values of n. Whence differentiating again, we have

$$2\frac{d\mu_{w+n}}{dn}=\frac{d\mu_{x+n}}{dn}+\frac{d\mu_{y+n}}{dn},$$

or, what is the same thing.

$$2\frac{d\mu_{w+n}}{dw}=\frac{d\mu_{x+n}}{dx}+\frac{d\mu_{y+n}}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for all values of n. Putting now $n=o$ in equations (1) and (2), we get

$$2\mu_w=\mu_x+\mu_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2\frac{d\mu_w}{dw}=\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Supposing now x and y to vary so that w remains constant, we get, by differentiation with respect to x,

$$\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy}\cdot\frac{dy}{dx}=o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{d^2\mu_x}{dx^2}+\frac{d^2\mu_y}{dy^2}\cdot\frac{dy}{dx}=o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which, by eliminating $\frac{dy}{dx}$, we get

$$\frac{\frac{d^2\mu_x}{dx^2}}{\frac{d\mu_x}{dx}} = \frac{\frac{d^2\mu_y}{dy^2}}{\frac{d\mu_y}{dy}} = k \text{ (say)} \quad . \quad . \quad . \quad . \quad (7)$$

since the equation holds for any values of x and y. From this by integrating we get

$$\log\frac{d\mu_x}{dx} = kx + l \text{ (say)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\frac{d\mu_x}{dx} = e^{kx+l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence integrating again,

$$\mu_x = m + \frac{1}{k}e^{kx+l} = m + \frac{1}{k}e^l . e^{kx} \quad . \quad . \quad . \quad . \quad (10)$$

Putting now $e^k = c$, $\frac{1}{k}e^l = \mathrm{B}$, and $m = \mathrm{A}$, we get the familiar expression

$$\mu_x = \mathrm{A} + \mathrm{B}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is equivalent to

$$\frac{d \log l_x}{dx} = -\mathrm{A} - \mathrm{B}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

whence by integration

$$\log l_x = p - \mathrm{A}_x - \frac{\mathrm{B}}{\log c}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where p is an arbitrary constant. Putting now $p = \log k$, $\mathrm{A} = -\log s$, and $\frac{\mathrm{B}}{\log c} = -\log g$, we get

$$\log l_x = \log k + x \log s + c^x \log g \quad . \quad . \quad . \quad . \quad (14)$$

or

$$l_x = k s^x g^{c^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is the well-known expression of Makeham's law. Thus we see that Makeham's is the most general law to which the principle of uniform seniority can be applied.

R. HENDERSON.

Ottawa,
3 *August* 1895.

[The proposition proved above was referred to by the late Mr. Woolhouse so long ago as 1870 (*J.I.A.*, xv, 402), in the following words, viz.: "It may further be stated that a rigid analytical proof might be given that Mr. Makeham's formula, which includes that of Gompertz, is the most general form of function possible to which a law of uniform seniority can in any way be applicable."—ED. *J.I.A.*]

ON APPROXIMATE VALUES OF THE FORCE OF MORTALITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The following extension of formula (21) on page 25 of the *Text-Book*, Part II, may perhaps be of some interest.

Writing $\frac{l_{-1}-l_{+1}}{l_0}=\text{A}$, $\frac{l_{-2}-l_{+2}}{l_0}=\text{B}$, $\frac{l_{-3}-l_{+3}}{l_0}=\text{C}$, and so on; we get the successive approximations to the value of μ, as follows:

(1) $\frac{\text{A}}{2}$

(2) $\frac{2}{3}\text{A}-\frac{1}{3.4}\text{B}$

(3) $\frac{3}{4}\text{A}-\frac{3}{4.5}\text{B}+\frac{1}{4.5.6}2\text{C}$

(4) $\frac{4}{5}\text{A}-\frac{6}{5.6}\text{B}+\frac{4}{5.6.7}2\text{C}-\frac{1}{5.6.7.8}2.3\text{D}$

(5) $\frac{5}{6}\text{A}-\frac{10}{6.7}\text{B}+\frac{10}{6.7.8}2\text{C}-\frac{5}{6.7.8.9}2.3\text{D}+\frac{1}{6.7.8.9.10}\times 2.3.4\text{E}.$

The first and second are those given in the *Text-Book* under formulas (22) and (21) respectively; and are correct to $\Delta^2 l_0$ and $\Delta^4 l_0$ respectively; the third and fourth were worked out by expressing μ, A, B, &c., in terms of the finite differences of l_0, and are correct to $\Delta^6 l_0$ and $\Delta^8 l_0$ respectively; the fifth was inferred by analogy and found to be correct to $\Delta^{10} l_0$. It will be noticed that the numerators of the coefficients of B, 2C, 2.3D, &c., are figurate numbers of the second, third, fourth, &c., orders. On making use of the third approximation instead of the second, the values of μ_4 and μ_5 derived from the table on page 494 of the *Text-Book*, Part II, were found to be ·01208 and ·01148, instead of ·01379 and ·01142 respectively, while from μ_6 to μ_9 inclusive there was no change. The second approximation is useful for accurately drawing tangents to curves of not higher than the fourth degree, and the further approximations could similarly be used for the curves of higher degrees.

I am, Sir,
Your obedient servant,
H. N. SHEPPARD.

3 *Chichester Road,*
London, W.,
6 *November* 1895.

[From the symmetrical form of the coefficients, we may assume that the general, or nth, approximation would be

$$\frac{n}{n+1}\text{A}-\frac{\frac{n.\overline{n-1}}{2}}{\overline{n+1}.\overline{n+2}}\text{B}+\frac{\frac{n.\overline{n-1}.\overline{n-2}}{3}}{\frac{\overline{n+1}.\overline{n+2}.\overline{n+3}}{\underline{|2}}}\text{C}-\&\text{c.},$$

and that this would be a correct approximation to the value of μ to $\Delta^{2n} l_0$.—ED. *J.I.A.*]

Syphilis as affecting Life Insurance Risks. Report by Dr. E. J. Marsh, *Medical Director of The Mutual Life Insurance Company of New York.* [*Reprinted by permission.*]

It has been hitherto the rule of the Mutual Life, under the advice of its medical examiners, to decline all applicants who give a history of having had syphilis. I do not know when the rule was first established, but it was probably in the early years of the company. This rule has been enforced almost uniformly, and, if there were any exceptions, they were certainly very rare. During this long period, however, our knowledge of the natural history of this disease has been greatly increased by careful and patient observations, and there has been corresponding improvement in the methods of treatment. Consequently, the question has been raised and discussed by medical men, as to whether, with this better knowledge and treatment, the old rule might not be too rigid, and whether it should not be either annulled or at least amended, and whether such exclusion of syphilitics from the benefits of insurance was not injurious to the company and unjust to the applicants. At the present time there is a strong preponderance of medical opinion that a syphilitic history is not to be considered an absolute bar to life insurance. As a matter of practice in the different life insurance companies the rule depends upon the personal experience, judgment, or prejudice of the medical advisers. I have now endeavoured to consider the question fairly and thoroughly, and to review the latest and best opinions of our medical authorities on the whole history of syphilis and its influence on the duration of the lives of those affected by it.

Syphilis begins as a local disease, arising from direct inoculation of the virus. Its first appearance is insignificant, but in a short time there is evidence of systemic infection. This is shown by certain peculiarities of the sore, glandular enlargement, fever, the appearance of characteristic spots or eruptions on the skin and mucous membrane, &c. Under appropriate treatment these manifestations may soon subside, but they are apt to recur at occasional intervals for a period of one, two, or more years, after which time they may disappear for ever, leaving no sign of their former existence in the individual. In a certain percentage of cases this is the history of syphilis, even when no medical treatment has been undergone, and, with a suitable treatment, it is the history in a very large proportion.

If this were the *entire* history, there would be no question of insurance, and the disease would be regarded as only one of the minor ills of life. But, unfortunately, in a certain number of cases more grave results are observed. Occasionally in the first year, but generally after a longer period of latency, the disease shows itself in more severe and deep-seated forms, ulcerations of the skin or mucous membrane, or affections of the bones, viscera or nervous system. These local affections show no tendency to self limitation, but continue to increase or spread unless checked by medical treatment.

The affections of the nervous system are the most dangerous, and at the same time, unfortunately, least amenable to treatment—some proving entirely incurable.

For while recent medical science has made progress in mitigating the severity of syphilis, it has, on the other hand, disclosed its manifestations and influence in the causation of diseases beyond what had formerly been suspected. Several serious nervous affections, as locomotor ataxia, general paralysis, and other diseases of the central nervous system, are now recognized as due to syphilis in a very large number of cases.

There is generally a period of latency between the so-called secondary symptoms and these tertiary and more severe lesions, and it is in this condition that a man will apply for insurance. The individual appears perfectly well, feels perfectly well, continues well for a period varying from a few months to very many years, so that he even forgets his early sickness, and then suddenly the disease shows itself again in the form of a tumour, an ulceration, a paralysis, or other nervous affection—diseases which undoubtedly often bring life to a premature close. These eventualities must not be forgotten or glossed over in considering the question of insurance.

After this brief sketch of the natural history of syphilis, two points come up for consideration.

1. The self-limitation or curability of syphilis.

In an absolute, strict sense, it can never be asserted or proved that syphilis is cured or ceases to affect an individual: and yet, in a practical common-sense view, it is often seen to be cured. If a man contracts syphilis in early life, exhibits secondary symptoms for a few years only, then lives past middle age and dies without any recurrence, in ordinary language he might reasonably be said to have been cured; but, on account of the long period of latency which is sometimes observed, the cure cannot positively be pronounced. Fournier states that he has observed tertiary syphilis first appearing fifty-five years after infection. Therefore, if our supposed case had lived a few years longer, the disease might have shown itself again. Syphilitic lesions certainly re-appear after a long period of apparent health, and a physician can never state positively that they will never re-appear, and pronounce his patient cured. It is in this view of the question that Gowers writes: "When we speak of the *cure* of a disease, we "mean that its essential element, that which lies behind its "symptoms and consequences, that which is the persistent cause "behind the transient effects—we mean that this is made to cease, is "ended once and for all as a morbific agent, so that it never again "disturbs the system. In this sense I believe it is literally correct to "say that we have no evidence that syphilis ever is or ever has been "cured." Dr. Keyes expresses himself strongly to the same effect: "Hence the difficulty of saying when syphilis has ended, or indeed "of deciding that it ever does end, since it so often modifies the "diathesis of the individual who has suffered from it. . . . . "Syphilis, once acquired, stamps its impress upon the individuality "of the patient and becomes a part of him, and no power on earth "in a given case can say when that impress disappears." And yet

it is the almost unanimous opinion of medical writers that the large majority of syphilitics do get well absolutely, entirely, and that it is only in a minority that these tertiary symptoms occur. Dr. Keyes writes again: "The probability of the disease in most cases, however, "is that its manifestations will disappear finally after a few years, "and this, under intelligent management, becomes almost a "certainty." While some cases of syphilis appear to be self-limited, and come to an apparent end without having been submitted to any medical treatment, this termination or cure becomes far more probable, and it extends to a far greater proportion of cases, when an appropriate and efficient treatment has been undergone. I find that this is made by all writers an essential pre-requisite before they will pronounce the probability of a patient's remaining well, and can recommend an affected individual either for marriage or for insurance.

2. In what proportion of cases of syphilis is the disease mild and of short duration, and in what proportion grave? There are no observations or statistics by which the relative numbers can be absolutely determined, and at best we can get only rough estimates made by those who have most experience in the treatment of this class of patients. These estimates range from 5 to 20 per-cent of severe (tertiary) cases, and it will not be far wrong to assume it as 10 per-cent—that is to say, according to these authorities, that 9 cases out of 10, all of which are submitted to a proper course of treatment, will remain well and free from all tertiary symptoms of the disease. One of the 10 will at some subsequent period develop other and more serious lesions which may compromise his life, although most of them are still amenable to treatment. This proper course of treatment means a medical treatment and management lasting from one to four or more years, as may be indicated for the individual case: and, in absence of this prolonged treatment, the number and severity of tertiary cases would be far greater.

In estimating the probability of the recurrence of a disease, the lapse of time is an element of great value. It is sometimes supposed that the tertiary symptoms are apt to be late in occurring, and that after the first outburst of the disease has subsided, there will generally be a long period of latency. But, systematic observation has shown that this is not correct: while this period of latency may be very prolonged, in the majority of cases the tertiary lesions appear within a few years. Fournier gives "the following statistics based on "2,395 cases in which the date of invasion of tertiarism, under all "forms of manifestation, could be determined exactly:

"During the 1st year	106 cases	
,, ,, 2nd ,, ...	227 ,,	
,, ,, 3rd ,, ...	256 ,,	
,, ,, 4th ,, ...	229 ,,	
,, ,, 5th ,, ...	205 ,,	
,, ,, 6th ,, ...	201 ,,	
Total within 6 years		1,224
From 6th to 10th year		499
From 10th to 20th year		543
Above 20 years		129
		2,395"

From this it appears that if the disease is to assume the tertiary form, it will do so in more than one-half the cases within 6 years, and in 72 per-cent within 10 years. In only 28 per-cent was the first occurrence delayed 10 years.

After this brief description of the nature and course of the disease, the original question may be considered. Can syphilitics be insured; and, if so, under what circumstances and conditions, and how can the risk be reduced to a minimum?

I cannot advise the abrogation of the present rule, but only such a modification as would allow of numerous exceptions in its practical working. The statement that a person had had syphilis should be looked upon as an impediment, but not as an absolute bar to life insurance. It is an impediment that might and ought to be cleared away by satisfactory explanation. There is a presumption of non-insurability, and the burden of proof for the removal of this presumption should rest upon the applicant.

It has been stated that the vast majority of syphilitics never have any lesions offering danger to life, provided that they have taken proper treatment, but that in a small minority of cases dangerous tertiary symptoms recur. The endeavour should be to select the good and reject the bad only. I think this might be accomplished by acting according to the following suggestions:

1. No case with the history of any primary venereal sore should be accepted until six months shall have elapsed after its first appearance. If, however, in the absence of all constitutional treatment, no other symptom, such as glandular enlargement, eruptions, mucous patches, may have appeared by this time, the applicant might be acceptable. If he has undergone any constitutional treatment, a further postponement of six months after the termination of such treatment is necessary.

2. No person with a history of syphilis is insurable until after a proper course of treatment and the lapse of at least six years from the date of infection.

3. No person can be accepted who may have any history or evidence of tertiary manifestations.

4. On the other hand, a person may be accepted who gives a history of constitutional syphilis, provided the original disease may not have been severe; that he shall have undergone a prolonged and satisfactory course of treatment, and a period of six years may have elapsed since the initial lesion, during the last two of which no relapses have appeared, and no tertiary symptoms at any time. I cannot advise any *rule* for the acceptance of persons who have not been submitted to proper treatment; such treatment is to be regarded not only as a great safety against future dangers, but also as an assurance that in the case of a possible re-appearance of new symptoms, medical advice would at once be sought and followed.

With regard to the time limit, I have fixed upon six years because the statistics of Fournier show that half the cases of tertiary syphilis occur during these years, and the probability of their

appearances diminishes rapidly after that period. Fournier does not approve of the marriage of syphilitics until four years have elapsed, and I think that for life insurance six years is not too short. If the period was extended to ten years, the danger would be very considerably less. Moreover, only persons who are up to the *full* standard of physique and health should be accepted. If there is any other partial disqualification, in addition to the syphilitic history, the applicant should be declined.

As a practical matter in dealing with applications for insurance, there is great difficulty in obtaining the information necessary to form any well-grounded judgment. The answers to the printed questions are short and meagre, and often even misleading. I would recommend, therefore, that in all cases a special communication be sent to the local examiner, with a request for such details as the company's medical officers desire. This history should come, if possible, from the attending physician, but, if this be impossible, the local examiner should certify that he is satisfied as to the accuracy of the applicant's statement. The report should give the date of the initial lesion, the kind, duration, and severity of the secondary symptoms and the presence or absence of tertiary lesions, the kind and duration of the treatment, and the name of the attending physician. In want of this specific information no insurance should be approved, but, when furnished, the company's medical adviser could generally form a correct judgment of the case. He could satisfy himself as to the *greater* or *lesser* probability of future danger, and while recognizing the impossibility of a certain prognosis, he could accept many applicants for the company without any more risk than he assumes daily in acting upon various other classes of cases.

13 *November* 1895.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On some methods of grouping Policies for the purpose of Valuation. *By* FREDERICK SCHOOLING, F.I.A., *Actuary of the Prudential Assurance Company.*

[Read before the Institute, 30 December 1895.]

IN dealing with the valuation of a large number of policies, the method of grouping is of very great importance, and although there is probably nothing new in any of the suggested methods, I venture to hope that a discussion of this subject will be of some utility.

For whole-life assurances, where the sum assured remains stationary throughout life, the ordinary method of grouping according to attained ages at the date of a valuation is perhaps the most usual and labour saving. In Friendly Society valuations the number of groups is sometimes still further reduced by using the central age of each group of five ages; in such valuations it is impossible to use dates of birth for the purpose of grouping according to nearest age, because the age at entry is the only information available.

It often happens, however, that the sum assured by industrial whole-life policies increases in amount after having been in force a certain number of years. This fact prevents the policies being grouped according to valuation ages in the ordinary way, because

the single premium to assure £1 over the whole of life cannot be used to find the value of the sum assured at each attained age at the date of a valuation, regardless of the duration of the policies.

Obviously some method of grouping must be adopted which gives effect to the duration of the policies.

It must be borne in mind that in the case of policies effected by means of weekly payments the amounts assured vary and not the premiums, and that a weekly premium of twopence assures just twice as much as a weekly premium of one penny, and so on. In a valuation of policies effected under these conditions, the method of grouping adopted which successfully coped with the difficulties of the increasing sums assured, was to value each year of issue separately and to consider each policy issued for a multiple of one penny premium to be as many penny policies as were contained in the multiple. For example, a policy at a weekly premium of sixpence would be reckoned, for the purpose of obtaining the liability only, as six policies at one penny.

In industrial assurance, policies on female lives are at least as numerous as those on male lives. The sexes were therefore valued separately. This necessitated a double set of tables, which were prepared in the following form, there being one table for each year of assurance.

Year of Assurance n.

Office Age at Entry x	$\frac{1}{2}+a_{x+n}$	Present Value of Sum Assured by Weekly Premium of One Penny	Net Annual Premium for Sum Assured by Weekly Premium of One Penny	Net Liability for One Penny Policy at Age $x+n$	Office Age at Entry x
1					1
2					2
3					3
4					4
&c.					&c.

There were eighty-four of these tables prepared for the forty-two years of assurance. These tables once constructed can be used at each successive valuation.

Before the results could be tabulated in above form, preparatory tables were calculated to obtain the various columns, the formulas used being:

If S be the original sum assured for a weekly premium of one penny, $S+\beta$ the sum assured after five years, and $S+2\beta$ the sum assured after ten years.

Value of Sum Assured by Weekly Premium of One Penny. Age at Entry x.

For year 0 of assurance $(1+i)^{\frac{1}{2}}\left\{\dfrac{SM_x+\frac{1}{2}\beta(M_{x+4}+M_{x+5})+\frac{1}{2}\beta(M_{x+9}+M_{x+10})}{D_x}\right\}$

„ 1 „ $(1+i)^{\frac{1}{2}}\left\{\dfrac{SM_{x+1}+\frac{1}{2}\beta(M_{x+4}+M_{x+5})+\frac{1}{2}\beta(M_{x+9}+M_{x+10})}{D_{x+1}}\right\}$

. . . .

„ 5 „ $(1+i)^{\frac{1}{2}}\left\{\dfrac{(S+\beta)M_{x+5}+\frac{1}{2}\beta(M_{x+9}+M_{x+10})}{D_{x+5}}\right\}$

„ 6 „ $(1+i)^{\frac{1}{2}}\left\{\dfrac{(S+\beta)M_{x+6}+\frac{1}{2}\beta(M_{x+9}+M_{x+10})}{D_{x+6}}\right\}$

. . . .

For year 10 and following years of assurance $(1+i)^{\frac{1}{2}}\left\{\dfrac{(S+2\beta)M_{x+n}}{D_{x+n}}\right\}$.

The constants $\frac{1}{2}\{M_{x+4}+M_{x+5}\}$ and $\frac{1}{2}(M_{x+9}+M_{x+10})$ were used instead of M_{x+5} and M_{x+10} because the valuation supposition adopted after investigation, was that all policies enter on the average in the middle of a calendar year at age $x-\frac{1}{2}$, attaining the exact age x at the end of year 0 of assurance. Thus, at the date of a valuation, there are four and a half years to run before the first increase in the sum assured takes place for year 0 of assurance, three and a half for year 1, and so on.

For years of assurance 10 and upwards the policies might have been grouped in valuation ages, because no further increase in the sum assured takes place. This, however, would have robbed the method suggested of many of its advantages.

The formula used to obtain the net annual premium for the sum assured by a weekly premium of one penny was

$$\frac{SM_x+\beta M_{x+5}+\beta M_{x+10}}{D_x}\div(\tfrac{1}{2}+a_x).$$

By multiplying the value of the sum assured and not the premium into $(1+i)^{\frac{1}{2}}$ the provision for the immediate payment of claims is thrown entirely upon the value of the sum assured. It is perhaps more strictly correct to operate upon a liability found upon the assumption that claims are paid at the end of the year of death, and thus increase the value of the future premiums as well as the value of the sum assured, but the method adopted brings out larger reserves, leaves the premium used untouched, and thus does not lessen the margin between it and the gross premium.

It will be seen that the full particulars required by the Board of Trade can be obtained from these valuation tables, given the equivalent number of one penny policies. The sum assured, by multiplying the sum assured by a weekly premium of one penny into the equivalent number of one penny policies. The gross annual premium is the number of equivalent penny policies multiplied by 52 and expressed in £. The net annual premium is found by multiplying the net premium required to cover the risk of a penny policy into the equivalent number of one penny policies. The value of the sum assured, by multiplying the equivalent number of one penny policies into the value of the sum assured by one penny policy. The value of the future gross and net premiums, by multiplying the annuity into the gross and net premiums respectively.

Although the value of the future net premiums had to be obtained to arrive at the net liability under each one penny policy, it was not tabulated, it being preferred to calculate the value of the future premiums in the manner described, because by thus doing both the calculation of the tables, and the liability, were checked. The liability at each group was found by the multiplication of the equivalent number of one penny policies into the net liability for one penny policy, and also by deducting the value of the future net premiums from the value of the sum assured, the two results agreeing exactly.

One advantage of this system of valuing is, that given the equivalent number of one penny policies standing for each age at entry in each year of issue, the liability can be found by one multiplication in each case.

The system lends itself naturally to a method of keeping valuation class books, because it is only necessary to record the number of policies at the various weekly premiums, no notice being taken of the sum assured. This fact minimises any chance of error.

The entries are made in the following form, which represents one page of a book in which is recorded the number of policies lapsed during the current year for one age at entry, and for one sex, in a particular year of issue. There being separate books for each year of issue, for lapses, revivals, and deaths; and in the case of the current year an additional book for policies issued.

Year of Issue 18 PRESENT OFFICE AGE MALES.

Number of Policies at Risk 31*st December*, 1893..

Office Age at Entry *Equivalent Number of One Penny Policies*

1894	WEEKLY PREMIUM OF LAPSED POLICIES																							
Week ending	One Penny	Two Pence	Three Pence	Four Pence	Five Pence	Six Pence	Seven Pence	Eight Pence	Nine Pence	Ten Pence	Eleven Pence	One Shilling												Total
Total																								
Equivalent Number of One Penny Policies																								

At the end of the year the facts derived from these books are tabulated as follows:

Year of Issue 18...... *Males.*

Office Age at Entry x	Office Age, 31 Dec., 1894 $x+n$	Equivalent Number of One Penny Policies							
		Lapsed	Deaths	Total Discontinuances	Revived	Net Decrease	At Risk, 31 December 1893	At Risk, 31 December 1894	Net Liability £

For the purpose of making the entries in the above books, the card system was adopted. The cards representing the lapses, deaths, revivals, and new business, being sorted and counted each week. The results being recorded on schedules preparatory to entering in the books. By means of these schedules the totals at each present office age are obtained, and also the totals for each year of issue. The sum of the totals at present office ages agreeing with the sum of the totals in years of issue, week by week. At the end of each year by means of a total book, the entries made in the class books are checked. For, the total book records the number of policies respectively lapsed, revived, died, and issued, both in present office ages, and in years of issue, and the summary of the class books must agree with these numbers.

The system of valuing annually in years of issue allows a very interesting and useful comparative return to be produced as follows:

Year of Issue 18—.

Age at Entry	1890		1891		1892		1893		1894	
x	Equivalent No. of One Penny Policies	Net Liability	Equivalent No. of One Penny Policies	Net Liability	Equivalent No. of One Penny Policies	Net Liability	Equivalent No. of One Penny Policies	Net Liability	Equivalent No. of One Penny Policies	Net Liability

This comparison acts as an additional safeguard against any large error creeping in, and is in itself interesting as showing how long a year's issue of policies has to be in force before the liability becomes stationary or decreasing.

We also have all materials to hand for preparing mortality tables either in the aggregate or after any given number of years of assurance, but as this subject does not come within the scope of this paper, I pass on to the consideration of the grouping of joint-life policies.

Joint Lives.

It will generally be conceded that where there are thousands, or tens of thousands of joint-life policies to value, some system of grouping must be adopted.

One system which has been used is to group the policies according to the nearest equivalent single age, using the annuity values for this purpose. That is, roughly assume that Gompertz's law of mortality holds throughout life.

This system is, however, open to objection: two lives may have the same equivalent integral single age for two consecutive years, it is therefore impossible to use net premiums according to the equivalent single age, as the ages do not progress in regular sequence. This prevents the grouping of policies at equivalent ages at entry, and necessitates the heavy work of using net premiums for the enormous number of combinations at the joint-life ages. Also it is a system which prevents the same valuation group being used from year to year, the policies

requiring re-grouping at every valuation. Thus it is altogether unfitted for a series of valuation class books.

Another method of grouping joint-life policies and one which has the sanction of the high authority of Mr. Ralph P. Hardy is thus described in the Board of Trade Returns. "The ordinary "Joint-Life Assurances were valued in groups, according to the "sum of the average ages attained by the lives on 31 December "1890, and on the same principles as the ordinary Whole-Term "Assurances."

In the sixth schedule the following table is given, which I give in full as it is explanatory of the system adopted.

Sum of Ages	Ages taken as	Sum of Ages	Ages taken as	Sum of Ages	Ages taken as	Sum of Ages	Ages taken as
48	26 and 22	79	42 and 37	107	56 and 51	134	69 and 65
49	27 ,, 22	80	42 ,, 38	108	56 ,, 52	135	70 ,, 65
51	28 ,, 23	81	43 ,, 38	109	57 ,, 52	136	70 ,, 66
53	29 ,, 24	82	43 ,, 39	110	57 ,, 53	137	71 ,, 66
55	30 ,, 25	83	44 ,, 39	111	58 ,, 53	138	71 ,, 67
56	30 ,, 26	84	44 ,, 40	112	58 ,, 54	139	72 ,, 67
57	31 ,, 26	85	45 ,, 40	113	59 ,, 54	140	72 ,, 68
58	31 ,, 27	86	45 ,, 41	114	59 ,, 55	141	73 ,, 68
59	32 ,, 27	87	46 ,, 41	115	60 ,, 55	142	73 ,, 69
60	32 ,, 28	88	46 ,, 42	116	60 ,, 56	143	74 ,, 69
61	33 ,, 28	89	47 ,, 42	117	61 ,, 56	144	74 ,, 70
62	33 ,, 29	90	47 ,, 43	118	61 ,, 57	145	75 ,, 70
63	34 ,, 29	91	48 ,, 43	119	62 ,, 57	146	75 ,, 71
64	34 ,, 30	92	48 ,, 44	120	62 ,, 58	147	76 ,, 71
65	35 ,, 30	93	49 ,, 44	121	63 ,, 58	148	76 ,, 72
66	35 ,, 31	94	49 ,, 45	122	63 ,, 59	149	77 ,, 72
67	36 ,, 31	95	50 ,, 45	123	64 ,, 59	150	77 ,, 73
68	36 ,, 32	96	50 ,, 46	124	64 ,, 60	151	78 ,, 73
69	37 ,, 32	97	51 ,, 46	125	65 ,, 60	152	78 ,, 74
70	37 ,, 33	98	51 ,, 47	126	65 ,, 61	153	79 ,, 74
71	38 ,, 33	99	42 ,, 47	127	66 ,, 61	154	79 ,, 75
72	38 ,, 34	100	52 ,, 48	128	66 ,, 62	155	80 ,, 75
73	39 ,, 34	101	53 ,, 48	129	67 ,, 62	156	80 ,, 76
74	39 ,, 35	102	53 ,, 49	130	67 ,, 63	157	81 ,, 76
75	40 ,, 35	103	54 ,, 49	131	68 ,, 63	158	81 ,, 77
76	40 ,, 36	104	54 ,, 50	132	68 ,, 64	159	82 ,, 77
77	41 ,, 36	105	55 ,, 50	133	69 ,, 64	160	82 ,, 78
78	41 ,, 37	106	55 ,, 51				

It will be noticed that the ages taken are always either four or five years apart. No doubt very careful investigation was made before deciding upon the ages taken.

This method is very simple in application. The policies can be grouped in "sums of ages" at entry, and the net premium obtained in these groups if thought desirable. Policies need not

be re-grouped at each successive valuation, the "sum of ages" increasing by two for each year of assurance.

Another method of grouping joint-life policies, and the one here advocated, is to group according to the equivalent equal ages, using the annuity-values for the purpose. This of course involves the assumption that Makeham's modification of Gompertz's law holds in the Mortality Table in question. In the *Text-Book*, Part II, page 256, it is pointed out that "a mortality "table constructed according to Makeham's modification of "Gompertz's law, such as the Mortality Table at the end of this "volume, affords great facility for calculating the values of joint-"life annuities."

If a distinction is made between male and female mortality—that is, if two distinct tables of mortality are used for the purpose of obtaining the commutation columns D_{xy} and N_{xy}, as is the case in the English Life Table No. 3, we have to assume that Makeham's modification of Gompertz's law holds, and also that the constant c in the one table equals the constant c in the other. Thus the expression becomes slightly altered when we are dealing with a combination of two different mortality tables. We have then

$$\mu_x = A + Bc^x$$

$$\mu_y = A' + B'c'^y$$

$$\mu_{xy} = A + A' + Bc^x + B'c'^y$$

say $$\mu_{ww} = A + A' + Bc^w + B'c'^w.$$

If we substitute here (w) (w) for (x) (y), we assume $Bc^w + B'c'^w = Bc^x + B'c'^y$.

This substitution will be justified if $\mu_{w+n:w+n}$ may be shown equal to $\mu_{x+n:y+n}$.

$$i.e., A + A' + Bc^{w+n} + B'c'^{w+n} = A + A' + Bc^{x+n} + B'c'^{y+n}.$$

Assume $c = c'$, then $c^n(Bc^w + B'c'^w) = c^n(Bc^x + B'c'^y)$, and therefore $\mu_{w+n:w+n} = \mu_{x+n:y+n}$ when two tables are concerned.

The value of c is fairly constant in all normal tables, and as the male and female tables of the English Life Table No. 3 are very similar in their general characters, no violence is done in assuming c to be of the same value in each. This point is referred to by Mr. Ryan, in his paper on "the late William Matthew Makeham" (*J.I.A.*, xxx, 7), where he collates the

results of some of the principal applications of Mr. Makeham's method.

Dr. Farr, in his introduction to the English Life Table, states "The rates of female mortality are in singular accordance with "those deducible from observations on males; the mortality of "females being slightly higher at the ages 10 to 35 than the "mortality of males at home in England."

It is also well known that the H^M Table coincides very closely with a table constructed according to Makeham's modification of Gompertz's law, and that the American experience table coincides exactly.

The following comparison [see page 311] between equivalent annuity-values at equal ages and the values at exact ages, practically shows that the one may be safely used in place of the other where the number of policies is large.

It will be seen from this comparison that the equivalent equal ages can be very readily computed; as the deduction from the older life to find the equivalent equal ages is the same in all cases where the sex of the older life, and the difference in ages, are both constant. Thus, at ages 32 and 27, the difference in ages being five years, and age 32 being the male life, the necessary deduction from age 32 is two, the equivalent equal ages being 30 and 30. The deduction from the older life is also two years for all combinations of ages where the difference is five years and the older life male. The same rule of course applies if the older life is female, the number of years to be deducted being different in some cases than when the male is the older life.

A table in the following form, which can be used without any technical knowledge whatever, is all that is necessary to enable the equivalent equal ages to be found.

English Life Table No. 3, 3 per-cent. Two Joint Lives, Male and Female.

Deduction from Older Life to find Equivalent Equal Ages			
Δ	Older Life Male	Δ	Older Life Female
1	0	1, 2	1
2, 3	1	3, 4	2
4, 5, 6	2	5, 6, 7	3
7, 8, 9	3	8, 9	4

English Life Table No. 3—Three per-cent.
Male and Female Joint Lives. The Male the elder.
Comparison of Annuity-Values.

Equal ages	$\mathbf{a}_{x:x}$	Difference in age 3 years	$\mathbf{a}_{x:x+3}$	Difference in age 5 years	$\mathbf{a}_{x:x+5}$	Difference in age 8 years	$\mathbf{a}_{x:x+8}$
30 : 30	16·358	28 : 31	16·405	27 : 32	16·349	25 : 33	16·344
31 : 31	16·150	29 : 32	16·197	28 : 33	16·139	26 : 34	16·130
32 : 32	15·938	30 : 33	15·985	29 : 34	15·925	27 : 35	15·914
33 : 33	15·722	31 : 34	15·769	30 : 35	15·706	28 : 36	15·693
34 : 34	15·502	32 : 35	15·548	31 : 36	15·484	29 : 37	15·468
35 : 35	15·276	33 : 36	15·324	32 : 37	15·257	30 : 38	15·239
36 : 36	15·046	34 : 37	15·094	33 : 38	15·026	31 : 39	15·007
37 : 37	14·812	35 : 38	14·860	34 : 39	14·790	32 : 40	14·770
38 : 38	14·572	36 : 39	14·622	35 : 40	14·550	33 : 41	14·528
39 : 39	14·328	37 : 40	14·379	36 : 41	14·305	34 : 42	14·283
40 : 40	14·078	38 : 41	14·131	37 : 42	14·057	35 : 43	14·033
41 : 41	13·824	39 : 42	13·879	38 : 43	13·803	36 : 44	13·778
42 : 42	13·565	40 : 43	13·622	39 : 44	13·545	37 : 45	13·520
43 : 43	13·301	41 : 44	13·360	40 : 45	13·282	38 : 46	13·257
44 : 44	13·032	42 : 45	13·094	41 : 46	13·015	39 : 47	12·989
45 : 45	12·757	43 : 46	12·823	42 : 47	12·743	40 : 48	12·717
46 : 46	12·478	44 : 47	12·546	43 : 48	12·466	41 : 49	12·440
47 : 47	12·193	45 : 48	12·265	44 : 49	12·185	42 : 50	12·159
48 : 48	11·902	46 : 49	11·979	45 : 50	11·898	43 : 51	11·873
49 : 49	11·606	47 : 50	11·687	46 : 51	11·606	44 : 52	11·588
50 : 50	11·304	48 : 51	11·390	47 : 52	11·315	45 : 53	11·300
51 : 51	10·995	49 : 52	11·093	48 : 53	11·020	46 : 54	11·007
52 : 52	10·685	50 : 53	10·792	49 : 54	10·721	47 : 55	10·710
53 : 53	10·370	51 : 54	10·485	50 : 55	10·416	48 : 56	10·409
54 : 54	10·048	52 : 55	10·173	51 : 56	10·107	49 : 57	10·104
55 : 55	9·734	53 : 56	9·855	52 : 57	9·792	50 : 58	9·795
56 : 56	9·420	54 : 57	9·530	53 : 58	9·472	51 : 59	9·481
57 : 57	9·106	55 : 58	9·214	54 : 59	9·146	52 : 60	9·164
58 : 58	8·792	56 : 59	8·898	55 : 60	8·829	53 : 61	8·843
59 : 59	8·479	57 : 60	8·583	56 : 61	8·513	54 : 62	8·519
60 : 60	8·167	58 : 61	8·270	57 : 62	8·200	55 : 63	8·204
61 : 61	7·857	59 : 62	7·959	58 : 63	7·889	56 : 64	7·892
62 : 62	7·549	60 : 63	7·650	59 : 64	7·582	57 : 65	7·585
63 : 63	7·245	61 : 64	7·345	60 : 65	7·278	58 : 66	7·282
64 : 64	6·945	62 : 65	7·044	61 : 66	6·979	59 : 67	6·984
65 : 65	6·650	63 : 66	6·748	62 : 67	6·685	60 : 68	6·692
66 : 66	6·361	64 : 67	6·458	63 : 68	6·397	61 : 69	6·406
67 : 67	6·079	65 : 68	6·174	64 : 69	6·116	62 : 70	6·127
68 : 68	5·805	66 : 69	5·898	65 : 70	5·842	63 : 71	5·856
69 : 69	5·538	67 : 70	5·629	66 : 71	5·577	64 : 72	5·592
70 : 70	5·280	68 : 71	5·369	67 : 72	5·320	65 : 73	5·337
71 : 71	5·031	69 : 72	5·118	68 : 73	5·071	66 : 74	5·090
72 : 72	4·792	70 : 73	4·876	69 : 74	4·832	67 : 75	4·853
73 : 73	4·562	71 : 74	4·644	70 : 75	4·603	68 : 76	4·625
74 : 74	4·343	72 : 75	4·422	71 : 76	4·383	69 : 77	4·406

Under this method of valuing according to equivalent equal ages, policies can either be grouped in years of issue and valued in the same manner as whole term single-life policies, by the equivalent number of one penny policies, or they can be valued in groups of valuation ages, with the sum assured, office and net premiums set out in the usual way.

The method can also be applied in all cases where large numbers of policies are involved, whether the premiums are payable yearly, half-yearly, quarterly, or weekly.

For the purpose of comparing the method of valuing by means of the "sum of the ages" method with the equivalent equal ages method I have made the following comparison of annuity-values—

Sum of Ages	Sum of Five Values			Sum of Ten Values		
	Correct Annuities	Equivalent Age Annuities	Sum of Ages Method $5a_{xy}$	Correct Annuities	Equivalent Age Annuities	Sum of Ages Method $10a_{xy}$
70	70·799	70·920	70·890	138·865	139·016	141·780
85	61·299	61·236	61·410	119·118	119·174	122·820
100	50·909	50·902	51·045	98·422	98·060	102·090
115	39·021	38·960	39·145	74·570	74·484	78·290

The ten annuity-values used for this comparison were those for the ten annuities with the shortest intervals between the ages. Thus, for "sum of ages 70" the annuities used were those for ages 35 and 35, 36 and 34, 37 and 33, 38 and 32, 39 and 31, 40 and 30, 41 and 29, 42 and 28, 43 and 27, and 44 and 26. The sum of five values being the sum of the first five with the shortest interval between the ages.

It will be noticed that the sum of the actual annuity-values are generally more nearly in accordance with the sum of the annuities for the equivalent equal ages, both at summations for five and ten years, than with five or ten times the annuity at the "ages taken as" under the sum of ages method.

Endowment Assurances.

The question of grouping endowment assurance policies for the purpose of valuation is one of growing importance, in consequence of the very large increase in the number of this class of policies of late years. I think it will be generally conceded that in any system of grouping endowment assurance

policies, the number of future payments is of more importance than the exact age.

In the valuation of policies effected by means of weekly premiums, where the sum assured is payable at a given age or previous death, the valuation assumption that all policies enter in the middle of a calendar year at age $x-\frac{1}{2}$ is approximately correct. So that at the date of a valuation on 31 December, according to the assumption, an exact age has been attained and policies have an exact number of years to run.

A method of grouping, probably the usual and best in such cases, is to have all policies payable at the same age as primary groups, subdividing each group into valuation ages, using $\left(\frac{1}{2}+\frac{a_{x+n:\overline{t-n-1}}+a_{x+n:t-n}}{2}\right)$ as the annuity for the purpose of finding the value of the future office and net premiums, where there are $\overline{t-n}$ years premiums still remaining to be paid and the valuation age is $x+n$. This method is applicable to policies effected by means of annual, half-yearly, or quarterly premiums, the valuation assumption being varied according as investigation shows it to be necessary, and each valuation age being further subdivided according to the number of future half-yearly or quarterly payments.

If the tables of a company provide for the payment of the endowment after a given number of years, instead of at a given age, the method of grouping described above involves enormous labour, as the policies become payable at any age from 35 to 70, and thus thirty-five distinct tables require valuation. In this case policies do not mature at an exact age, nor according to the valuation assumption if entries are spread evenly throughout the year, at the end of a year, but in the middle. Different valuation factors are therefore necessary.

Another method of grouping endowment assurances, whether payable at a given age or in a given number of years, and whether effected by means of weekly or annual premiums, is to have each number of future payments as a separate group, using an average valuation age at each group for the purpose of valuing the sum assured and the office and net premiums. This method is, however, more particularly applicable in the case of policies payable in a given number of years, as in this case much labour is saved.

For the purposes of a test valuation these groups are all that is necessary, but in an actual valuation the following method of

grouping before dividing into number of future payments can be adopted; as many groups being formed as is found desirable.

Let all policies maturing under age n form group 1, all maturing between ages n and m form group 2, and so on. The general rule being, that the older the ages at maturity the smaller the groups, because it is in these groups that the question of mortality becomes of increasing importance. To find the average or valuation age for each group, it is well to operate upon the sum assured, so that due weight is given to policies for large amounts. This can be done as follows: for each number of future payments, multiply the present office age into the total amounts assured at that age, then the sum of these products divided by the sum of the amounts assured, gives the valuation age for the particular group in question.

The following schedule illustrates the method:

Group A.—*Policies maturing under Age n. Number of Future Payments t.*

Present Office Age (1)	Sums Assured (2)	Sums Assured × Present Office Age (3)
x		
$x+1$		
$x+2$		
&c.		

Then $\dfrac{\Sigma \text{ col. } 3}{\Sigma \text{ col. } 2} = \text{average or valuation age.}$

The particulars for valuation can be scheduled as follows, one schedule being required for each group.

Group A.—*Policies maturing under Age n.*

No. of Future Payments (1)	Average Valuation Age (2)	Number of Policies (3)	Original Sums Assured (4)	Bonus Additions (5)	Annual Reductions of Premium (6)	Full Annual Premiums: Office (7)	Full Annual Premiums: Net (8)
t							
$t-\frac{1}{4}$							
$t-\frac{1}{2}$							
$t-\frac{3}{4}$							
$t-1$							
&c.							

This method can be adapted, by means of cards and valuation class-books, to the keeping of permanent records in groups, as readily as the method of grouping according to valuation ages. It should be noted, however, that although the number of future payments decreases by one each year, the average valuation age does not necessarily increase by one, owing chiefly to the introduction of new lives at younger average ages. It is therefore better, following the plan of obtaining the valuation age previously indicated, in recording the various deductions and additions at each separate number of future payments, to also record the product of the actual age attained into the sum assured. So that at the end of the year the average valuation age for each group can be computed by merely adding the difference between the additions and deductions to the last year's totals.

Policies effected under tables other than whole-life, joint-life, or endowment assurance, are generally not of sufficient number to warrant a system of class-books, and are more readily valued each year directly from the cards, which can be kept in tables ready for valuation at any moment.

Discussion.

The President (Mr. A. J. Finlaison, C.B.) said Mr. Schooling had usefully discussed the nature of the devices which had to be employed in the Valuation of an assurance company with a large number of policies. The system proposed for the valuation of joint-life assurances would well repay careful study.

Mr. R. P. Hardy said there could be no doubt that a good set of valuation books and forms, especially making provision for the information sometimes required in auxiliary enquiries, was of the highest importance, and no time would be misspent in perfecting a piece of machinery by which they had to perform one of the most responsible functions of their professional life. In these days, in addition to knowing the reserve and the surplus, they wished to ascertain how the latter had arisen, and to trace the profit to its source. For that reason it appeared to him that the original facts should always be grouped round the central point of the valuation age. It might be interesting as a matter of history to mention that one of the earliest groupings for valuation purposes in the form of debtor and creditor account was due to a former actuary of the Eagle, some sixty years ago. The present author was more fortunately situated than most of them, working as he did upon a scale sufficient to allow full play to the various forces, and to enable real groupings to be made with a minimum of disturbance of the averages. On the other hand his materials were so heavy that he was obliged to settle his course at the outset. If he might venture upon a suggestion he would

ask the author to consider whether it would not be both instructive and interesting to value the bonus increment of the policies separately, and to see how it compared with the profit from lapses, or from an improvement in the general rate of mortality. The grouping of endowment assurances presented a difficulty owing to their recent rapid extension and their great variety. His own opinion for some time past had tended to grouping under the number of premiums to be paid, as it was found that on the liability side of the account some very convenient trial checks could be made when this method was adopted. He threw out these few remarks in order that a discussion upon this interesting subject might be well started, but he hoped that the author would not take their brevity as any measure of his sense of the importance of the methods disclosed, or of their educational value to their younger brethren.

Mr. E. Woods said one of the most important duties of an actuary in connection with the valuation of the liabilities of an office was to test the accuracy of the mathematical work, and there was no safeguard so sure as an annual valuation with class books going on from year to year. In this connection he thought that Mr. T. G. C. Browne's method of distribution had hardly been sufficiently considered from the point of view of its value as an independent check to the ordinary valuation work. He (Mr. Woods) had used it in two different offices for about fifteen years, and he found that the surplus disclosed by it was always almost identical with that shown by the ordinary valuation method. With regard to the valuation of joint life policies, in the majority of ordinary offices the number of such assurances was very small, so that it was hardly worth the trouble of dividing them into groups, however necessary it might be to do so in the case of industrial offices. The question of the valuation of endowment assurance policies, however, was of practical importance to all offices, whether ordinary or industrial, as the number of such policies issued, and the increase in their aggregate amount, in the last few years had been enormous. In 1888 the proportion of endowment assurances to other policies, in ordinary companies, was only 13·4 per-cent, while now it was about 28·6 per-cent—the total number of policies being 370,308, and the total sum assured £68,982,280. Perhaps he might be allowed to interpolate the remark that in 1849 the late Samuel Brown estimated the total number of policies of all kinds in force in the United Kingdom to be 300,000, for £150,000,000. He had found this information in the account of a meeting of English and foreign representatives of assurance interests held in the rooms of the Institute in July, 1851, which seemed to have been of the nature of an international conference. With regard to the classification of endowment assurances he would venture to utter a note of warning. It was a subject which must be looked into very carefully, for he knew of more than one company where the results of a classified valuation differed from those of a detailed one.

Mr. T. G. Ackland said industrial assurance was very special in character, and did not appeal to each one present from personal knowledge and experience in the same way that it did to Mr. Schooling. He was, however, disposed to agree with Mr. Hardy

that after ten years' duration some of the methods might have been simplified, but doubtless there were reasons why that was not done in the present case. The author referred to provision for the immediate payment of claims, and he (Mr. Ackland) entirely agreed that in some cases it was desirable that that provision should be based upon the value of the sum assured rather than upon the value of the net liability, as in the latter case there might arise considerable disturbances in the relation between the values of the net and gross premiums. Passing to the consideration of joint-life assurances, he must confess that the interesting method of grouping these by the sums of the ages was entirely new to him. There were other methods by which joint-life assurances could be dealt with in groups. One was to schedule such policies under the value of the annuity at the present age, so that for instance, if the present value of the annuity was 10·9, all joint policies having this as the nearest value of the annuity at their respective present ages would be grouped together. This plan had the disadvantage that it had to be done separately on every occasion, and one or other of the methods indicated in the paper would certainly be preferable in this respect. Passing on to the important matter of endowment assurances, he noticed that while averaging the ages in groups, the author dealt with much minuteness with the question of the number of premium payments, and this did not appear to be altogether consistent. It seemed to be somewhat unnecessary that the future payment should be set out for every quarter of a year, which certainly involved a great deal of labour. The whole question of grouping endowment assurances was very interesting, and at the same time very difficult; and for this reason, among others, that these assurances came before them in practically three forms—firstly, the old-fashioned form in which the sum assured was payable on a stated birthday with premiums payable throughout the whole term; secondly, where the sum assured was payable on a birthday but with a limited number of premiums; and thirdly, the form employed largely on the Continent, where the sum assured was payable at the end of a definite number of years, and with premiums also limited to that term. In each of these three cases the true expressions for the single premium, the annuity, and the net premium, all differed; so that it seemed to be almost hopeless to attempt any method of grouping that would be entirely satisfactory to meet all these cases.

Mr. H. W. Manly said that having been engaged in the valuation of industrial insurance societies he knew from experience the enormous mass of figures that had to be dealt with, and the necessity for classification in some form. He thought the classification made by the author of the ordinary whole-life contracts an admirable one, the system enabling the actuary to trace the whole of the business transacted in any one year through each succeeding course of years. When the schedules were in working order the extra labour involved in the valuation was small, and the advantage of maintaining the schedules was much greater than the saving that would result from grouping all the policies together after ten years for mere purposes of valuation. As to the valuation of joint-life assurances he thought that many persons were unaware of

the enormous number of these policies issued by industrial offices. More than 200,000 were issued by one society alone, which showed the absolute necessity of grouping them in some form or other. He had adopted a somewhat similar grouping to that mentioned, and found it acted exceedingly well. If there was a deviation from the exact value it was always on the right side, for the value brought out by such a system of grouping was always larger than that obtained by exact methods. The classification of endowment assurances was a matter he had himself closely studied, and without laying claim to originality he believed he was one of the pioneers in using a system of classification for these assurances. He had always adopted the system of classifying according to the year of maturity. In offices doing both large and small assurances, it was desirable that some weight should be given to the amounts assured at each existing age, and for that purpose he multiplied the present age by the sum assured at that age dividing the sum of those products by the total of the sums assured. He then valued the premiums by the annuity for $(n-1)$ years, and the sums assured by the corresponding values of the reversions, the necessary corrections being afterwards made. An objection might be raised that the ages in any one grouping would be so extensive and the values of the annuities at those ages so different that the system must prove a failure. On examination, however, this would prove not to be the case, and giving due consideration to the amount assured at each age, it would be found on comparison that a valuation by this method of grouping would be close enough for all practical purposes.

Mr. J. A. Robertson, referring to the grouping of endowment assurances according to the number of future premiums payable, said he had made a good many trials, and found that, so far as without-profit assurances were concerned, the results agreed very well, but in the with-profit assurances the differences were very considerable, sometimes as much as 1 per-cent. The same results had been experienced in at least one other office, and he would like to learn from the author whether he could suggest any modification of the method in applying it to with-profit policies.

Mr. J. H. Barnes said Mr. Hardy had reminded them that the author had an advantage over most actuaries in the large numbers he had to deal with, and when an empirical method of grouping such as that suggested for endowment assurances was employed, the question of numbers might have an important bearing on results. He therefore ventured to submit a comparison of the results in what might probably be regarded as a normal case. He recently had to value a group of 613 endowment assurances under which a gross amount of nearly a quarter of a million was assured, with tontine bonuses of nearly £12,000, and net premiums of £9,126. The business extended over about 20 years, the annual new business showing, as a rule, a fair rate of increase. The future duration of the policies ranged from 1 to 40 years, some maturing as young as 40, others at such an advanced age as 75; the bulk at the central ages, 55 and 60. The policies were originally valued in groups under the endowment ages, which of course gave a true result. As a check, he grouped them under the "assumed year of

maturity" (found by adding the number of future payments + 1 to the year of valuation) without regard to the endowment age. The differences were as follows: Value of sums assured (gross) *less* by £178 (·12 per-cent); the value of tontine bonuses *more* by £117 (1·72 per-cent); the value of the net premiums *more* by £325 (·31 per-cent); and the final net liability (after correcting for re-assurances, &c.) *less* by £388 (·78 per-cent). Those results were most satisfactory, and indicated that where the circumstances of a "model office" prevailed the method might be relied on, even when the grouping was made in a single series. He would like to refer to another method of grouping endowment assurances which had been successfully employed and which reduced the labour of computation almost as much as the foregoing, while the results were *certain* to be very near the truth. Before starting the scheduling a table must be constructed showing the corresponding whole-life age for each attained age at each quinquennial endowment age. Thus at age 30 the annuity for death or age 60 was 16·853 at 3 per-cent, and that corresponded most nearly to that for age 41, whole-life. The policies were thus scheduled as if they were whole-life policies at the assumed valuation ages, the correct net premium for the endowment assurance being filled in. That method could of course be relied on for an actual valuation, and would be found to save a vast amount of work in computation. It does not however lend itself to an annual valuation as the equivalent ages do not increase by unity for each corresponding increase in the real age. In constructing the preliminary table the "corrected" annuities that would be employed in the valuation must of course be used.

The President in proposing a vote of thanks to the author, said most of the systems they had heard explained for the classification of endowment assurance policies were essentially alike. Mr. Schooling's system had been commented on to some extent on the assumption that he employed a large number of groups, but there was nothing in his paper necessarily leading to that conclusion. He thought some of the paragraphs might with advantage be expanded, and that some further explanation might be afforded on one or two of the points dealt with.

Mr. Schooling, in reply, said Mr. Hardy's suggestion as to valuing the increments separately would form a valuable check on the computation. Mr. Manly thought he had used unnecessary groups in the valuation of endowment assurances, but it must be remembered that he was dealing with policies numbering some hundreds of thousands. He ought to state that the method of valuing joint-life assurances was first suggested to him by the late Mr. Griffin, and also that he had had in these matters the invaluable advice of Mr. Bailey.

On the system of bonus distribution to policyholders as a percentage per annum for the Valuation period on the sum assured, or on the sum assured and existing bonuses, considered in relation to some recent influences on Life Assurance Finance. By H. W. Andras, F.I.A., *Secretary of the University Life Assurance Society.*

[Read before the Institute, 27 January 1896.]

Many valuable papers have been read before this Institute in recent years on the subject of the sources of surplus, on various methods of bonus distribution and on kindred subjects, notably those of Mr. Sunderland in 1887 and 1890 (*J.I.A.*, xxvi, 357, and xxviii, 257); Mr. Rothery in 1892 (*J.I.A.*, xxx, 131); Mr. G. F. Hardy in 1894 (*J.I.A.*, xxxi, 261); and Mr. G. J. Lidstone in 1895 (*J.I.A.*, xxxii, 73); which are, in my opinion, contributions to actuarial science of the utmost value and models of concise language and elegant demonstration for the actuarial student.

It seems, however, a long time since a paper was read here relating specially to what I may perhaps be allowed to call the staple form of bonus distribution in this country, as described in the title of this paper, either the "simple" or "compound" form of it being adopted by about half the number of ordinary Life offices and by most of the ordinary branches of industrial offices of the United Kingdom. As far back as 1857 a paper was read by Dr. Sprague (*J.I.A.*, vii, 61), in which he treated of the mode of calculating the rates of premiums to be charged and the conditions to be observed to render such methods of distribution equitable, and since then the system has been alluded to in papers on the subject of "surplus" by Mr. Pattison (*J.I.A.*, ix, 341), and Mr. Meikle (*J.I.A.*, xi, 251), and in the more recent papers referred to.

Developments in the construction of office premiums, based upon what are termed "Select" Tables of Mortality, and loaded more in accordance with modern practice, the higher rates of expenditure of some companies in recent years, the considerable fall in the average rate of interest realized by Life offices during the last twenty years, and finally, the severe competition between offices at the present day, lead me to think it desirable to draw attention to and invite discussion amongst the members of this Institute on some effects of these influences on this method of bonus distribution, although of course they are by no means confined to offices declaring bonuses on this system.

It is first desirable to state the precise nature of the system of bonus under consideration, and the various forms of it now in use amongst Life Assurance Companies in this country.

Leaving out the ordinary branches of Industrial Life Offices, foreign and colonial Life offices doing business in this country, Life offices in course of transfer to other companies, assessment Societies and Societies for special objects not coming within the range of ordinary Life Assurance business, and small societies with an insignificant business, I estimate from the Board of Trade returns relating to Life Assurance Companies, that out of 66 ordinary Life offices 33 adopt the system of bonus under consideration, of which 18 distribute on the "simple", or as it is more usually called the "uniform" reversionary bonus plan, as an equal percentage on the sum assured only for the valuation period, and fifteen on the "compound" reversionary bonus plan as a percentage per annum for the valuation period on the sum assured and reversionary bonuses attaching. All these 33 offices, except two, have been established over 30 years, and of these two, one (a "uniform" bonus office) has been established 14 years, and the other (a "compound" bonus office) 12 years. Of the 18 "uniform" bonus offices one has not yet declared a bonus on that system, but has announced the intention of doing so at the next distribution of profits, and of the 15 "compound" offices, two are in a similar position: these three offices being all important old-established offices, I think the course adopted by them indicates a decided preference for this system of bonus allotment at the present day, and more especially for the compound form of it. All these offices, except two, distribute bonuses quinquennially, and these two (compound bonus offices) distribute septennially.*

The reversionary bonus allotted is calculated as a percentage per annum on the sum assured only, or on the sum assured and existing bonus additions for the number of years the policy has been in force, or for the number of years' premiums due and paid during the valuation period. The bonus allotted at the first distribution after the policy is effected, in respect of the period between the date of commencement of the assurance and the first distribution, in some offices vests at once, no matter how short a time the policy may have been in force, but as a rule there is a period from the date of assurance which must elapse

* The ordinary branch of a large industrial office distributes annually on the uniform bonus plan, but I know of no other office distributing bonuses *annually* under this system.

before any allotted bonus vests, a reserve being made at the valuation for the deferred bonuses allotted to policies which have not yet qualified to participate.

From answers to Question 8, in the fifth schedule of the Board of Trade Returns, under "The Life Assurance Companies Act of 1870", I gather the following:

The time during which a policy must be in force in order to entitle it to share in the profits.

Uniform Bonus Offices.

No. of Offices.	
1	At the first division after issue.
1	Nine months.
1	A policy must have been one or more years in force, but policies less than a year in force participate in the next division for the full number of annual premiums paid.
1	When a policy has been over one year in force.
1	When more than one annual premium has been paid.
2	When a policy has been two years in force, but in one case at the first division the policy participates in respect of each year in force except the first.
2	On payment of the third year's premium.
2	When a policy has been three years in force.
1	A policy begins to share in profits on payment of the fourth year's premium, but the full rate of bonus is not declared (only a rate graduating up to the full rate) on policies on which less than eight year's premiums have been paid.
4	The policy must have been in force five years.
1	Five years' premiums must have been paid.
1	Six years' premiums must have been paid.
Total 18	

Compound Bonus Offices.

No. of Offices.	
6	From the date of issue.
1	After payment of one year's premium.
1	When three years' premiums have been paid.
7	When the policy has been five years in force.
Total 15	

It is evident that the compound bonus offices have a greater tendency than the uniform bonus offices to allow the first bonus allotted to vest at once, however brief may have been the duration of the policy before the first distribution, and this I think is due to the nature of the compound bonus plan, commencing with a small reversionary bonus and increasing to a larger one as the policy duration extends. Possibly the practice is intended to meet the view that a policyholder's representatives are entitled to a bonus even if death takes place in the very early years of assurance if a participating rate has been paid, but to meet this view effectually, it would be necessary to give an intermediate bonus in case of death in the interval between the date of assurance and the first distribution: such a practice is very exceptional, but I hear that it has existed: if the scale of office premiums is constructed with a view to the practice, and the premiums in other respects are ample as regards basis and loading for the normal conditions of the office, there can be little objection to it, but scales of office premiums, if formed with any reference to this system of distribution are, I believe, usually constructed as regards bonus loading on the assumption that a policy is taken out at the commencement of a quinquennium, and that the first bonus allotment takes place five years from the date of assurance, with no interim bonus in the meantime. On the other hand, the increasing number of offices having "discounted bonus" scales of premiums may lead to the provision in new scales of with-profit premiums for an intermediate bonus in case of death between the date of assurance and the first distribution; of course, the equivalent reduction for the discount of bonuses vesting after five years can be as readily calculated as that for immediate bonuses, but policyholders assuring on the full premium scale may not understand this, and think themselves unfairly treated in comparison with the "discounted bonus" policyholders (who get an immediate advantage) in not having their bonuses vested immediately on the first declaration and an intermediate bonus in case of death previously. At any rate, the extra amount required in the bonus-loading for such an extension of bonus privileges would be small.

There are, however, many reasons, in my opinion, why a bonus should not vest until a policy has been some years in force; for example, the heavy initial expense of a policy, the small interest profit on a policy only a few years in force, and the modern disposition to heavy reserves, even on policies of brief duration.

After the first quinquennial distribution an interim reversionary bonus now usually attaches. This interim bonus comes out of the profits of the current quinquennium and any undivided surplus carried forward from the previous division, so that it may be said to be an anticipation of a portion of the surplus out of which the next quinquennial bonus is declared, and to a certain extent diminishes the quinquennial bonus.

Of the "uniform bonus" offices only three do not, so far as I can ascertain, give interim bonuses in case of death between two distributions—all the "compound" offices give them. The interim bonuses are usually payable in case of death only, the percentage being often considerably less than the rate at the last distribution. Some make a stipulation that three years' premiums shall have been paid, or that the policy shall have been in force three, five, or even ten years before it can be entitled to interim bonus. Some "compound" offices give a percentage per annum *on the sum assured only* for each year completed, or for each annual premium due and paid since the last bonus declaration.

In at least one "uniform bonus" office and one "compound bonus" office the *cash value* of the interim bonus payable at death is given on surrender of the policy between two distributions.

The following varieties of interim bonuses occur in "compound bonus" offices:

An interim bonus at the full rate of the last distribution on all policies becoming claims between two divisions. Calculated on the sum assured and existing bonuses.

Sixteen-seventeenths of the rate declared at the last division after the policy has been at least five years in force. Calculated on the sum assured and existing bonuses.

Three-fourths of the rate declared at the last division. Calculated on the sum assured and existing bonuses.

Fifteen-sixteenths of the rate declared at the last division. Calculated on the sum assured only.

After participating policies have been in force five years they are entitled to the *cash value* of the same rate of reversionary bonus as that declared at the preceding division of profits for each annual premium due and received.

There is a decided tendency towards the somewhat recent practice of declaring a prospective interim bonus for each year

entered upon in the quinquennium, at a rate equal to or bearing a large proportion to that of the last quinquennial division, whether on the sum assured only, or on the sum assured and existing bonuses.

It is unusual for the interim bonus to vest each year as an addition capable of being surrendered to the office, but there are exceptions, and I can see no objection to the practice. In that case, if interim bonuses are given in case of death between the date of assurance and the first distribution, as well as between two subsequent distributions, at the full rate of the last bonus, whether calculated on the sum assured or on the sum assured and existing bonuses, practically annual bonuses are allotted, although they may be only declared quinquennially, but in the case of the compound bonuses the compounding only takes place every five years. Were the compounding to take place annually, at annual declarations, if the same rate of bonus were maintained and no bonuses surrendered, the sum assured and bonuses would increase annually in Geometrical Progression—a rate of increase not within practicable limits. Under the uniform bonus plan for the same rate of bonus the sum assured and bonuses increase in Arithmetical Progression, assuming no bonuses surrendered. Between the two systems we have the compound reversionary bonus of practice, the sum assured and bonuses increasing quinquennially in Geometrical Progression, if the same rate of bonus is maintained, and, during each five-year period being temporarily increased in Arithmetical Progression for five years only by the interim bonus, assuming no bonuses surrendered.

The following are the formulæ for premiums loaded for "uniform bonus" or "compound bonus," π'_x denoting the annual premium loaded for a uniform bonus and π''_x that for a compound bonus. The formula for the annual premium for an assurance of 1 and a uniform reversionary bonus of b vesting on the completion of each year from the date of assurance is

$$\pi'_x = \frac{M_x + bR_{x-1}}{N_{x-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if the bonus is for each year entered upon, the formula would be

$$\pi'_x = \frac{M_x + bR_x}{N_{x-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the first five years' bonuses of b per annum only vested when the

policy had been five years in force, and thereafter an annual vested bonus addition of b for each year entered upon, the formula would be,

$$\pi'_x = \frac{M_x + 5bM_{x+5} + bR_{x+5}}{N_{x-1}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is also the formula for an assurance of 1 with a quinquennial reversionary bonus of b per annum for the five years, and a prospective interim bonus of b per annum in case of death in the interval between two quinquennial distributions, for each year entered upon since the last distribution, for

$$\left.\begin{array}{l} 5bM_{x+5} + 5bM_{x+10} + \&c. \\ + b(R_{x+5} - R_{x+10} - 5M_{x+10}) \\ + b(R_{x+10} - R_{x+15} - 5M_{x+15}) \\ + \&c. \end{array}\right\} = 5bM_{x+5} + bR_{x+5}$$

The annual premium for an assurance of 1, and an annual reversionary bonus at the rate of b per unit per annum on the total sum assured and bonuses at the beginning of a quinquennium for each year completed in the quinquennium is as follows:

$$\pi''_x = \frac{M_x + \Sigma b}{N_{x-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where Σb is an abbreviation for

$$b(R_{x+1} - R_{x+6}) + b(1+5b)(R_{x+6} - R_{x+11}) \\ + b(1+5b)^2(R_{x+11} - R_{x+16}) + \&c.$$

If the annual bonus is for each year entered upon, the expression Σb becomes

$$b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) \\ + b(1+5b)^2(R_{x+10} - R_{x+15}) + \&c.,$$

and

$$\pi''_x = \frac{1}{N_{x-1}}[M_x + b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) \\ + b(1+5b)^2(R_{x+10} - R_{x+15}) + \&c. \quad . \quad . \quad (5)$$

The sum assured and bonuses together form a series of uniformly and annually increasing assurances for each five years of assurance, the annual increase being arrested at the end of each five years, and the rate of annual increase for the ensuing five years being at the rate calculated on the sum assured and bonuses existing at the commencement of the quinquennium. Perhaps the following

scheme will best explain the system of bonus, and the formula is self-evident from the scheme.

TABLE I.

	Annual Bonus	Total Bonuses	Sum Assured and Bonuses
1st year	b	b	$1+b$
2nd ,,	b	$2b$	$1+2b$
3rd ,,	b	$3b$	$1+3b$
4th ,,	b	$4b$	$1+4b$
5th ,,	b	$5b$	$1+5b$
6th ,,	$b(1+5b)$	$5b+b(1+5b)$	$(1+5b)+b(1+5b)$
7th ,,	$b(1+5b)$	$5b+2b(1+5b)$	$(1+5b)+2b(1+5b)$
8th ,,	$b(1+5b)$	$5b+3b(1+5b)$	$(1+5b)+3b(1+5b)$
9th ,,	$b(1+5b)$	$5b+4b(1+5b)$	$(1+5b)+4b(1+5b)$
10th ,,	$b(1+5b)$	$5b+5b(1+5b)$	$(1+5b)+5b(1+5b)=(1+5b)^2$
&c.	&c.	&c.	&c.

The form of uniform reversionary bonus at the present day is a quinquennial uniform bonus of b per unit per annum for the five years, calculated on the sum assured, and a prospective interim bonus in case of death between two quinquennial allotments of bonus at a somewhat less rate than b, calculated also on the sum assured for each year entered upon since the last bonus declaration.

Formula (3) represents the premium loaded for this form of bonus where the interim bonus is at the same rate as the quinquennial bonus, the form in which it is required for calculating premiums loaded for a minimum rate of bonus.

The modern form of compound reversionary bonus is a quinquennial compound bonus of b per unit per annum for the five years on the sum assured and existing bonuses, and a prospective interim bonus in case of death (or on surrender of the policy) between two quinquennial allotments of bonus at nearly the same (or even the same) rate b, and calculated on the sum assured and existing bonuses for each year entered upon since the last bonus declaration.

The following formula represents the premium loaded for this form of bonus where the interim bonus is at the same rate as the quinquennial bonus:

$$\pi''_x = \frac{1}{N_{x-1}}\left\{\begin{array}{l} M_x - M_{x+5} \\ + (1+5b)(M_{x+5}-M_{x+10}) + (1+5b)^2 \\ \qquad\qquad (M_{x+10}-M_{x+15}) + \&c. \\ + b[(1+5b)(R_{x+5}-R_{x+10}-5M_{x+10}) \\ + (1+5b)^2(R_{x+10}-R_{x+15}-5M_{x+15}) + \&c.] \end{array}\right\} \quad (6)$$

If an interim bonus of b per unit per annum for each year entered upon be given in case of death between the date of assurance and the first quinquennial division of profits, the following expression must be added to each side of the equation:

$$\frac{b}{N_{x-1}}(R_x - R_{x+5} - 5M_{x+5})$$

then the whole expression on the right-hand side of the equation reduces, and we have

$$\pi''_x = \frac{1}{N_{x-1}}\{M_x + b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) + \&c.$$

which is the same as formula (5), and the scheme in Table I would represent the new reversionary bonus, the total bonuses, and the sum assured and bonuses in each year from the date of assurance.

The above formulæ assume the interim bonus to be at the same rate as the quinquennial bonus, the form in which they are required for the calculation of office premiums loaded for a minimum rate of bonus. If the interim bonus be at a less rate than the quinquennial bonus, as we have seen is frequently the case in practice, let b be the rate of quinquennial bonus, and b' the rate of interim bonus for each year entered upon in the quinquennium, then we have the following more general formulæ.

UNIFORM REVERSIONARY BONUS.

$$\pi'_x = \frac{1}{N_{x-1}}\{M_x + 5b(M_{x+5} + M_{x+10} + \&c.) \\ + b'R_x - 5b'(M_{x+5} + M_{x+10} + \&c.)\} \quad . \quad (7)$$

which, if $b = b'$, reduces to formula (2); and if there be no interim bonus between the date of assurance and the first quinquennial division we must deduct $b'(R_x - R_{x+5} - 5M_{x+5})$ from each side of the equation, and we obtain

$$\pi'_x = \frac{1}{N_{x-1}}\{M_x + 5b(M_{x+5} + M_{x+10} + \&c.) \\ + b'R_{x+5} - 5b'(M_{x+10} + \&c.)\} \quad . \quad (8)$$

which if $b = b'$ reduces to formula (3); and if there be no interim bonus at all but only quinquennial bonuses,

$$\pi'_x = \frac{1}{N_{x-1}}\{M_x + 5b(M_{x+5} + M_{x+10} + \&c.)\} \quad . \quad . \quad (9)$$

Compound Reversionary Bonus.

$$\pi''_x = \frac{1}{N_{x-1}} \left\{ \begin{array}{l} M_x - M_{x+5} + b'(R_x - R_{x+5} - 5M_{x+5}) \\ \quad + (1+5b)(M_{x+5} - M_{x+10}) + b'(1+5b) \\ \qquad\qquad\qquad (R_{x+5} - R_{x+10} - 5M_{x+10}) \\ \quad + (1+5b)^2(M_{x+10} - M_{x+15}) + b'(1+5b)^2 \\ \qquad\qquad\qquad (R_{x+10} - R_{x+15} - 5M_{x+15}) \\ \quad + \qquad \&c. \qquad + \qquad \&c. \end{array} \right\} \quad (10)$$

which if $b=b'$ reduces to formula (5), and if there be no interim bonus between the date of assurance and the first quinquennial division deducting $b'(R_x - R_{x+5} - 5M_{x+5})$ from each side of the equation we obtain

$$\pi''_x = \frac{1}{N_{x-1}} \left\{ \begin{array}{l} M_x - M_{x+5} \\ \quad + (1+5b)(M_{x+5} - M_{x+10}) + b'(1+5b) \\ \qquad\qquad\qquad (R_{x+5} - R_{x+10} - 5M_{x+10}) \\ \quad + (1+5b)^2(M_{x+10} - M_{x+15}) + b'(1+5b)^2 \\ \qquad\qquad\qquad (R_{x+10} - R_{x+15} - 5M_{x+15}) \\ \quad + \qquad \&c. \qquad + \qquad \&c. \end{array} \right\} \quad (11)$$

which if $b=b'$ becomes formula (6), and if there be no interim bonuses at all,

$$\pi''_x = \frac{1}{N_{x-1}} \{ (M_x - M_{x+5}) + (1+5b)(M_{x+5} - M_{x+10}) \\ + (1+5b)^2(M_{x+10} - M_{x+15}) + \&c. \}$$

which reduces to

$$\pi''_x = \frac{1}{N_{x-1}} [M_x + 5b(M_{x+5} + (1+5b)M_{x+10} \\ + (1+5b)^2 M_{x+10} + \&c.)] \quad . \quad . \quad (12)$$

These formulæ, I think, sufficiently explain themselves, the expressions being simply the premiums for a series of temporary assurances for five years of the sums assured and bonuses at the commencement of each quinquennium, and, if there be interim bonuses, including the premiums for a series of temporary uniformly increasing assurances of the interim bonus for each five years. If we deduct ϖ_x (the pure premium) from each side of the equations for the loaded premium we get the equations for the bonus-loading, and it will then be observed that the expressions for the bonus-loading in the case of the "uniform reversionary bonus" formulæ, for the cases in which the quinquennial and

interim bonus rate are the same, are homogeneous with regard to b, but in the case of the "compound reversionary bonus" formulæ this is not so, so that having calculated the bonus-loadings in the case of the "uniform" plan at the rate of one per-cent per annum, those at any other rate are proportional, but this is not the case with "compound reversionary" bonus-loadings, *e.g.*, the bonus-loading for a compound reversionary bonus of two per-cent per annum is not twice that for a one per-cent compound reversionary bonus.

Formulæ (3) and (6) are those used subsequently in this paper for the calculation of premiums loaded for a simple or compound bonus.

Having stated the various forms of this bonus system and the formulæ for calculating the premiums loaded for profit or, by deducting ϖ_x, the bonus-loadings themselves, or by substituting D_x for N_{x-1} in the denominators of the expression for the bonus-loadings the *present values* of the bonus-loadings, I now proceed to the main purpose of this paper, namely, the effect on this system of bonus of some recent influences on Life Assurance finance.

It seems to me that the most important influences affecting bonus prospects generally, placing them as nearly as possible in the order of their importance, are (1) The rates of premium charged. (2) The system and basis of valuation. (3) The difference between the average rate of interest earned on the funds and the rate assumed in the valuation. (4) The strain of the expenses and commission on the income. (5) The rate of mortality experienced as compared with that expected. (6) Profits or losses upon realization of investments. (7) Profit from non-participating business. (8) Profits from surrenders or lapses. (9) In proprietary offices the proportion of the divisible surplus allotted to policyholders.

As regards the rates of premiums charged, it has been urged to be of special importance in the construction of office premiums for use in offices having this system of bonus that their loading for profit should be calculated with a view to this system. Dr. Sprague said (*J.I.A.*, vii, 63): "It is impossible to do justice "without considering the two subjects of the loading of the net "premium and the method of division of profits in the very "closest connection." The object of this is to secure equity in the bonus distribution as between entrants of different ages, and the necessity for the practice is incidental to all bonus systems by

which the bonus is declared in reversion independently of the cash surplus arising from the premium after payment for the risk under the sum assured and the expenses assessed upon the policy. The usual minimum rate of bonus adopted for such profit loadings is one per-cent per annum, and it is a matter of interest to examine the existing office premiums to see whether, assuming a minimum rate of expenditure, a mortality experience coinciding with that expected by standard tables, and an assumed average rate of interest realized, they are equitable, as between persons assuring at different ages and receiving a simple or compound reversionary bonus of one per-cent per annum for the valuation period and an interim bonus at the same rate in case of death between two distributions. To accomplish this, it is desirable to have a standard scale of office premiums based on pure premiums calculated by the assumed average rate of interest realized, and a table of mortality derived from the experience of assured lives generally, loaded for the particular form of bonus (simple or compound) for a bonus of one per-cent per annum, and on the same basis of mortality and interest as the pure premium, and further loaded for a minimum rate of expenditure and commission.

For standard office premiums for this purpose I have adopted in this paper 4 per-cent as the interest basis, inasmuch as the only standard scale of premiums now existing is based on that rate of interest, and comparatively few offices reach that standard. Furthermore, it is a rate of interest which is still realized by offices skilfully managed as regards their investments. It is quite possible that having regard to the fall in the average rate of interest realized by life offices a lower interest-basis, namely $3\frac{3}{4}$ per-cent, should enter into the calculation of standard office premiums, but at present hardly a single office could reach a $3\frac{3}{4}$ per-cent scale of premiums loaded for profit and expenses as hereinafter stated. The prevailing opinion amongst actuaries seems to be that the true table of mortality upon which to base a premium for any age is what is termed a "select" table of mortality derived from the experienced mortality of lives assured at the same age at entry. The H^M Institute of Actuaries' table, I need hardly mention here, is a mixed table, the rate of mortality at any given age being the rate for lives who have attained that age but who have been on the books of the offices for various periods, and hence the mortality of recently-selected lives is mixed with that of lives more or less past the influence of medical selection, and the table cannot be said to be a true table for the purpose. The "select" mortality

table based on the Institute of Actuaries' experience, used for the formation of office premiums, is that so generously published for the use of actuaries by Dr. Sprague, who has also suggested office premiums for whole-life assurances without-profits (*J.I.A.*, xxii, 396) based on the formula

$$\left\{\pi_{[x]} + \frac{\cdot 01}{\mathbf{a}_{[x]}} + \cdot 00125\right\} 1\cdot 075,$$

where $\pi_{[x]}$ is Sprague's Select Mortality Table 4 per-cent premium at age at entry (x) for a £1 whole-life assurance, $\frac{\cdot 01}{\mathbf{a}_{[x]}}$ the provision by the same table of mortality and rate of interest for the initial commission of £1 per-cent on the sum assured distributed over life, ·00125 the constant for expenses of management, and 1·075 the factor for the percentage loading for renewal commission and fluctuations and payment to shareholders or participating policyholders for their guarantee of a fixed assurance at a low fixed premium.

With-profit rates have been calculated on a similar basis and loaded in a similar way except that to $\pi_{[x]}$ is added the loading for a "simple" or "compound" bonus and for the final percentage loading the factor 1·05 is substituted for 1·075 (inasmuch as the loading for shareholders' or members' guarantee is not required) to provide for renewal commission and fluctuations.

The resulting annual premiums per £100 are here given:

	Age 20			Age 30			Age 40			Age 50			Age 60		
	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*
Sprague's "Select" Office Premiums without-profits	1	13	9	2	0	9	2	14	10	3	19	3	6	3	10
Ditto, with-profits, loaded for a simple bonus	2	0	7	2	8	9	3	4	1	4	9	7	6	14	5
Ditto, with-profits, loaded for a compound bonus	2	2	1	2	10	3	3	5	5	4	10	9	6	15	5

If the mortality basis be the H^M Institute of Actuaries' Table, the rate of interest four per-cent, and the loading for profit, expenses, &c., as above, the resulting premiums will be as follows:

	Age 20			Age 30			Age 40			Age 50			Age 60		
	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*
"H^M" Office Premiums without-profits . .	1	10	7	1	19	9	2	14	7	4	0	6	6	7	2
Ditto, with-profits, loaded for a simple bonus	1	17	4	2	7	8	3	3	9	4	10	8	6	18	2
Ditto, with-profits, loaded for a compound bonus	1	19	0	2	9	3	3	5	2	4	11	11	6	19	1

In his "select life" tables Dr. Sprague gives a table of the net annual premiums necessary to provide at four per-cent interest an assurance of one with a compound bonus of five per-cent at the end of each five years from the date of assurance, and an interim bonus of one per-cent per annum from the end of the last quinquennial period till the date of death, the formula used being

$$\pi''_{(x)} = \frac{1}{N_{x-1}} \left\{ \begin{array}{l} M_{(x)} - M_{x+5} \\ + 1\cdot05(M_{x+5} - M_{x+10}) + (1\cdot05)^2(M_{x+10} - M_{x+15}) + \&c. \\ + \cdot01[1\cdot05(R_{x+5} - R_{x+10} - 5M_{x+10}) \\ \qquad + (1\cdot05)^2(R_{x+10} - R_{x+15} - 5M_{x+15} + \&c.)] \end{array} \right\}$$

The corresponding formula for the simple bonus is

$$\pi'_{(x)} = \frac{M_{(x)} + \cdot05M_{x+5} + \cdot01R_{x+5}}{N_{(x-1)}}.$$

These formulæ assume that a policy is taken out at the commencement of a quinquennium, which is the usual assumption in making such calculations.

Dr. Sprague's method of loading for expenses and commission is, in my opinion, the one which best reflects the conditions of practice. Since 1881, when his suggested office premiums were published, the system of giving a commission of £1 per-cent on the sum assured in respect of the first year's premium and $2\frac{1}{2}$ per-cent on renewal premiums, either on with or without-profit whole-life and endowment assurance business, has become general; perhaps the minimum constant for expenses is nearer 3*s.* than 2*s.* 6*d.* at the present date, even in the most economically managed offices, but the percentage loading in the formula is rather more than is required for the $2\frac{1}{2}$ per-cent renewal commission, so that the loading, as it stands, is, perhaps, sufficient for a "standard" premium at present.

The scale of office premiums resulting from the "Select" mortality basis and four per-cent interest, with the loading as described, may be said to be the only *standard* now existing, but a few offices only of the "uniform reversionary" and "compound reversionary" class reach that standard at the principal assurance ages (from 20 to 40), most of them approximating more closely to the H^M Institute of Actuaries' office premiums loaded in a similar way. If, therefore, the "Select" Table is the true standard, the majority of offices appear to be giving too much bonus to the assured at the younger ages of entry relatively to that allotted to lives assuring at the older ages, having regard to the premiums charged, unless

there is some compensatory element in the way of mortality profit in the younger class of entrants. At about 40, it will be observed that the office premium by the "Select" Table and the "H^M" Table cross one another, the lives above that age paying more premium under the "H^M" basis than under the "Select" basis, showing the disadvantage to the older entrants in the matter of bonus in offices charging premiums similar to the H^M office premiums.

As an illustration of the office premiums charged, I give here the average and the highest and lowest premiums of the offices under the uniform and compound reversionary bonus systems at various ages at entry, for comparison with those based on the "Select" and "H^M" Tables.

18 "*Uniform Reversionary Bonus*" *Offices*.

Whole-Life, with Profits, Sum Assured £100	Age 20			Age 30			Age 40			Age 50			Age 60		
	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*
Average Annual Premiums	1	18	1	2	8	9	3	4	6	4	10	8	7	0	6
Highest Annual Premiums	2	1	5	2	10	9	3	5	11	4	14	3	7	11	7
Lowest Annual Premiums	1	15	8	2	6	4	3	2	4	4	6	10	6	7	4
"Select" Office Annual Premiums	2	0	7	2	8	9	3	4	1	4	9	7	6	14	5
"H^M" Office Annual Premiums	1	17	4	2	7	8	3	3	9	4	10	8	6	18	2

15 "*Compound Reversionary Bonus*" *Offices*.

Whole-Life, with Profits, Sum Assured £100	Age 20			Age 30			Age 40			Age 50			Age 60		
	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*
Average Annual Premiums	1	19	7	2	9	4	3	4	9	4	10	3	6	18	7
Highest Annual Premiums	2	2	1	2	11	9	3	6	3	4	12	4	7	4	9
Lowest Annual Premiums	1	17	0	2	6	11	3	2	1	4	7	5	6	14	3
"Select" Office Annual Premiums	2	2	1	2	10	3	3	5	5	4	10	9	6	15	5
"H^M" Office Annual Premiums	1	19	0	2	9	3	3	5	3	4	11	11	6	19	1

It will be observed that the "uniform reversionary bonus" offices have average premiums above 30 higher than the "select" office premiums, and under 30 higher than the "H^M" office premiums.

In the case of the "compound reversionary bonus" offices, the average premiums from 20 to 40 are well under the "select" office premiums but very near to the "H^M" office premiums.

In both cases at age 60 the average premiums are considerably higher than the "select" office premiums, but average premiums do not give us much information, and further investigation is necessary.

The following statement gives for each age at entry the number of offices having annual premiums equal to, in excess, or in defect of the "Select" and "H^M" 4 per-cent office premiums loaded by Sprague's method.

TABLE II.

18 UNIFORM REVERSIONARY BONUS OFFICES.

Age at Entry	"SELECT" 4 PER-CENT OFFICE PREMIUM			"H^M" 4 PER-CENT OFFICE PREMIUM		
	Number of Offices			Number of Offices		
	Equal	In excess	In defect	Equal	In excess	In defect
20	...	1	17	...	14	4
30	2	10	6	...	15	3
40	1	13	4	...	14	4
50	...	13	5	...	9	9
60	...	16	2	...	14	4

15 COMPOUND REVERSIONARY BONUS OFFICES.

Age at Entry	"SELECT" 4 PER-CENT OFFICE PREMIUM			"H^M" 4 PER-CENT OFFICE PREMIUM		
	Number of Offices			Number of Offices		
	Equal	In excess	In defect	Equal	In excess	In defect
20	3	...	12	...	8	7
30	1	3	11	1	6	8
40	1	4	10	...	5	10
50	2	3	10	1	1	13
60	2	11	2	...	6	9

These are very interesting statistics, and indicate in a rough way the relations of the offices to the "Select" and the "H^M" 4 per-cent office premiums, but they do not show the extent of the excess or defect from these premiums at the various ages at entry. Some of the office premiums, although in defect of one or other of these premiums, may be so little in defect at some or all ages as to be for all practical purposes the same: others may have been loaded for a less bonus than one per-cent per annum, and therefore be constructed on sound lines, although not as high as office premiums providing a one per-cent bonus. In order, therefore, to measure the extent of the excess or defect from these assumed standards at each age at entry in the offices with this system of bonus, I have taken out the following differences between the actual annual office whole-life with-profit premiums at ages 20, 30, 40, 50, and 60 at entry, and the "Select" and "H^M" 4 per-cent office premiums for those ages, the upper difference relating to the "Select" Table, and the lower to the "H^M" Table, the sign (+) signifying in excess, and the sign (—) in defect.

TABLE III.—18 UNIFORM REVERSIONARY BONUS OFFICES.
Amount in Excess (+) or Defect (−) of the "Select" or "H^M" 4 per-cent Office Premiums.

	Age at Entry 20	30	40	50	60
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
"Select" 4 per-cent Office Premiums . .	2 0 7	2 8 9	3 4 1	4 9 7	6 14 5
"H^M" 4 per-cent Office Premiums . .	1 17 4	2 7 8	3 3 9	4 10 8	6 18 2
"Select" 4 per-cent Loading for Bonus of 5 per-cent every 5 years and interim Bonus of 1 per-cent per annum . . .	0 7 3	0 8 6	0 10 0	0 11 7	0 12 11
"H^M" 4 per-cent do. .	0 7 2	0 8 5	0 9 11	0 11 6	0 12 10
	s. d.	s. d.	s. d.	s. d.	s. d.
(1)	− 4 11 − 1 8	− 2 2 − 1 1	− 0 8 − 0 4	+ 2 0 + 0 11	+ 7 1 + 3 4
(2)	− 4 9 − 1 6	− 1 11 − 0 10	− 1 9 − 1 5	− 2 9 − 3 10	+ 4 6 + 0 9
(3)	− 4 6 − 1 3	− 2 5 − 1 4	− 1 5 − 1 1	+ 2 7 + 1 6	+17 2 +13 5
(4)	− 3 7 − 0 4	− 0 3 + 0 10	− 0 7 − 0 3	+ 0 11 − 0 2	− 0 5 − 4 2
(5)	− 3 1 + 0 2	− 0 1 + 1 0	+ 0 7 + 0 11	+ 0 11 + 0 2	+10 1 + 6 4
(6)	− 2 11 + 0 4	+ 0 6 + 1 7	+ 1 5 + 1 9	+ 1 8 + 0 7	+11 6 + 7 9
(7)	− 2 8 + 0 7	+ 0 2 + 1 3	+ 0 4 + 0 8	− 0 7 − 1 8	+ 6 7 + 2 10
(8)	− 2 7 + 0 8	+ 1 3 + 2 4	+ 0 4 + 1 3	+ 0 5 − 0 8	+ 5 7 + 1 10
(9)	− 2 6 + 0 9	+ 0 3 + 1 4	+ 1 11 + 2 3	+ 4 8 + 3 7	+12 9 + 9 0
(10)	− 2 6 + 0 9	+ 0 1 + 1 2	+ 1 10 + 2 2	+ 2 4 + 1 3	+ 5 7 + 1 10
(11)	− 2 6 + 0 9	+ 0 1 + 1 0	+ 0 9 + 1 1	+ 2 9 + 1 8	+ 7 1 + 3 4
(12)	− 2 2 + 1 1	equal + 1 1	+ 0 5 + 0 9	+ 1 3 + 0 2	+ 6 4 + 2 7
(13)	− 1 11 + 1 4	+ 1 3 + 2 4	+ 1 3 + 1 7	+ 3 9 + 2 8	+ 7 11 + 4 2
(14)	− 1 10 + 1 5	+ 0 2 + 1 3	+ 0 8 + 1 0	+ 2 0 + 0 11	+ 4 9 + 1 0
(15)	− 1 3 + 2 0	+ 1 0 + 2 1	equal + 0 4	− 1 4 − 2 5	+ 6 10 + 3 1
(16)	− 1 2 + 2 1	equal + 1 1	+ 0 5 + 0 9	− 1 1 − 2 2	+ 1 4 − 2 5
(17)	− 1 1 + 2 2	+ 0 8 + 1 9	+ 0 5 + 0 9	+ 0 10 − 0 3	+ 2 1 − 1 8
(18)	+ 0 10 + 4 1	+ 2 0 + 3 1	+ 0 6 + 0 10	− 2 1 − 3 10	− 7 1 −10 10

Table IV.

15 Compound Reversionary Bonus Offices.

Amount in Excess (+) or Defect (−) of the "Select" or "H^M" 4 per-cent Office Premiums.

	Age at Entry 20	30	40	50	60
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
"Select" 4 per-cent Office Premiums . .	2 2 1	2 10 3	3 5 5	4 10 9	6 15 5
"H^M" 4 per-cent Office Premiums	1 19 0	2 9 3	3 5 3	4 11 11	6 19 1
"Select" 4 per-cent loading for bonus of 5 per-cent every five years and interim bonus of 1 per-cent per annum . . .	0 8 8	0 10 0	0 11 4	0 12 8	0 13 9
"H^M" 4 per-cent do. .	0 8 7	0 9 11	0 11 3	0 12 7	0 13 8
	s. d.	s. d.	s. d.	s. d.	s. d.
(1)	− 5 1	− 1 10	− 0 11	− 0 11	+ 0 11
	− 2 0	− 0 10	− 0 9	− 2 1	− 2 9
(2)	− 4 9	+ 1 5	− 0 6	− 0 3	+ 9 4
	− 1 8	− 0 5	− 0 4	− 1 5	+ 5 8
(3)	− 4 9	− 1 6	− 0 6	− 0 3	+ 9 4
	− 1 8	− 0 6	− 0 4	− 1 5	+ 5 8
(4)	− 4 7	− 1 11	− 1 5	− 0 11	+ 0 7
	− 1 6	− 0 11	− 1 3	− 2 1	− 3 1
(5)	− 3 11	− 0 5	+ 0 8	+ 1 2	+ 0 9
	− 0 10	+ 0 7	+ 0 10	equal	− 2 11
(6)	− 3 4	− 3 4	− 3 4	− 3 4	− 1 2
	− 0 3	− 2 4	− 3 2	− 4 6	− 4 10
(7)	− 3 4	− 1 9	− 1 8	− 0 11	+ 1 5
	− 0 3	− 0 9	− 1 6	− 2 1	− 2 3
(8)	− 2 0	− 1 1	− 0 6	+ 1 7	+ 6 11
	+ 1 1	− 0 1	− 0 4	+ 0 5	+ 3 3
(9)	− 1 10	− 1 6	− 1 0	equal	+ 1 8
	+ 1 3	− 0 6	− 0 10	− 1 2	− 2 0
(10)	− 1 9	+ 0 6	+ 0 6	equal	+ 4 0
	+ 1 4	+ 1 6	+ 0 8	− 1 2	+ 0 4
(11)	− 1 7	− 0 9	− 0 3	+ 0 2	equal
	+ 1 6	+ 0 3	− 0 1	− 1 0	− 3 8
(12)	− 0 2	− 1 0	− 1 10	− 2 1	− 1 2
	+ 2 11	equal	− 1 8	− 3 3	− 4 10
(13)	equal	+ 1 6	+ 0 10	− 0 8	+ 4 7
	+ 3 1	+ 2 6	+ 1 0	− 1 10	+ 0 11
(14)	equal	equal	equal	equal	equal
	+ 3 1	+ 1 0	+ 0 2	− 1 2	− 3 8
(15)	equal	+ 1 6	+ 0 10	− 0 2	+ 9 4
	+ 3 1	+ 2 6	+ 1 0	− 1 4	+ 5 8

The general conclusions I draw from these statistics, taking the "Select" 4 per-cent office premiums as my standard, are—

(1.) That relatively too much bonus is given in many offices to the class of persons assuring between 20 and 30, and too little to the class of persons assuring over 50, having regard to the premiums paid by each class.

The figures show unmistakably the influence of the H^M Mixed Table at ages 20 to 30 and over 50, in the former case depressing the office premiums *below* the "Select" standard, and in the latter case increasing them to a more considerable extent above the "Select" standard.

In a few cases, usually of office premiums based on the Carlisle Table, the former practice of loading pure premiums with a heavy percentage, instead of a bonus loading with a constant and a small percentage, may have had a somewhat similar effect to the H^M Table. For example, Carlisle 3 per-cent premiums with a loading of 25 per-cent (at all ages) are charged by some offices for "compound bonus" office premiums, and the result is a very low premium at age at entry 20, and a very high premium at age at entry 60. At age 40 the resulting premium is very nearly the same as the "Select" 4 per-cent office premium, and at that age the percentage loading amounts to 13*s*. added to the Carlisle 3 per-cent pure premium. If the constant 13*s*. had been added to the Carlisle 3 per-cent pure premiums at all ages, having regard to the resulting premium, a "compound" bonus of one per-cent would have been too little for lives under 40 and too much for lives over that age, illustrating the view that the system of loading by a constant gives relatively too little bonus to the young entrants, and too much to lives entering beyond middle age.

(2.) That the amounts by which the office premiums of some offices are in defect at the ages 20 to 30 bear too large a proportion to the cost of a one per-cent bonus to be neglected, if that is the minimum rate of bonus expected or assumed in the premiums charged. It is quite possible that an office charging office premiums, based on a Table showing a light mortality at the younger ages, such as that of the H^M mixed Table may argue that as its mortality experience is as good or even better than that

shown by that Table at similar ages, it is justified in basing its office premiums on the "H^M" Table rather than on the "Select" Table. Possibly in a "Class" office selecting its lives from a class of the community having a greater longevity than the average of assured persons it would be justifiable to make some allowance for mortality profit in calculating the premiums to be charged, but an office assuring the general public would have to experience a light mortality for a very long period and with great regularity before it could with safety anticipate mortality profit by a deduction from the premiums charged, and it might then be almost considered a class office making a specially careful selection of risks. It appears to me that an office assuring lives from the general public must charge a premium at any given age based on the mortality experience of average assured lives who became assured at that age, and let the accidental mortality profit of any period be divided as bonus.

(3.) With regard to the high rates charged by some offices at the older ages in comparison with the "Select" office premiums, I think we can say very little until we see the results of the new Experience: it is quite possible that the increased mortality amongst the older lives, from the ravages of influenza in the latter years of the Experience, may have the effect of slightly raising the new "Select" premiums above the present "Select" premiums, but I can hardly think they are likely to be increased to the present high figures of the H^M premiums.

Of the offices charging premiums materially less than the H^M 4 per-cent premiums at the younger ages, it is probable that one or two of them intentionally loaded for less than a one per-cent bonus in constructing their premiums. Perhaps, also, several of the offices in the "Compound" class loaded their premiums for a "Uniform" bonus instead of a "Compound" bonus, and trusted to interest-profit to produce the "bonus on bonus" results.

It is evident, therefore, that so far as the mortality basis for the construction of premiums is concerned, although the

"Battle of the Standards" has been waged as between "Select" and "mixed" Tables now for at least fourteen years, it has not yet been fought out, and probably will not be until the coming Experience takes the field; but in the meantime there is another influence affecting the basis of the office premiums, which may have the effect of raising office premiums constructed on sound principles at all ages at entry, namely, the fall in the effective rate of interest earned on the funds of offices. At present the basis of the office premiums I have taken as "standard" premiums involves a rate of interest of 4 per-cent, and it is seen that, notwithstanding this, they are higher than the existing with-profit office premiums of many offices, at the younger and principal assuring ages. But there are some people who deny that any scale of office premiums should be based on a higher rate of interest than $3\frac{1}{2}$ or 3 per-cent; they make no stipulations, however, as to the loading calculation, and they simply object to the higher rate of interest, and are unconcerned as to the method of loading; such views are sometimes met by adopting some special system of loading, and yet the final result is much the same as Dr. Sprague's "Select" office premiums, calculated at 4 per-cent, and with a system of loading more in accordance with practice; for example, the following comparison [see p. 341] of premiums on various bases and with various methods of loading is interesting.

It is easy enough, by changing the system of loading, to satisfy the requirements of persons demanding a 3 per-cent or $3\frac{1}{2}$ per-cent basis and yet arrive at similar office premiums to these calculated on a 4 per-cent basis with a loading more in accordance with actual facts, but it is more satisfactory to base office premiums for use with these systems of bonus on true Tables of Mortality and Interest, and a system of loading more in accordance with facts. Before true Tables of Mortality were published, derived from the experience of a large number of assured lives, probably unsatisfactory and somewhat rough methods of loading prevailed, of which traces still remain. Tables of office premiums were evolved in very curious ways from the old existing data; for example, a Table of office whole-life with-profit premiums was derived by deducting 5 per-cent from the Northampton 3 per-cent premiums for the principal assurance ages up to nearly 50, and thereafter graduating into the Northampton 3 per-cent premiums. The resulting premiums at ages under 50 are slightly higher than the "Select" 4 per-cent

TABLE V.

Annual Office Premiums for £100.

	Age 20			Age 30			Age 40			Age 50			Age 60		
	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
"Select" 4 per-cent premium for a compound bonus . . .	2	2	1	2	10	3	3	5	5	4	10	9	6	15	5
H^M 3½ per-cent "Select" premium + "Simple" bonus loading + ·125 + $\frac{1}{15}$th	2	2	4	2	10	9	3	6	3	4	11	9	6	16	6
or															
"Select" 4 per-cent premium for a simple bonus	2	0	7	2	8	9	3	4	1	4	9	7	6	14	5
"Select" 3 per-cent + 5s. + $\frac{1}{5}$th	2	0	10	2	8	4	3	3	4	4	8	10	6	11	10
or															
H^M 4 per-cent premiums loaded for a compound bonus, and expenses and commission by Sprague's method	1	19	0	2	9	3	3	5	2	4	11	11	6	19	1
H^M 3½ per-cent + simple bonus loading + ·125 + $\frac{1}{15}$th . . .	1	19	2	2	9	9	3	6	2	4	12	9	6	19	11
or															
H^M 4 per-cent loaded for simple bonus expenses and commission by Sprague's method	1	17	4	2	7	8	3	3	9	4	10	8	6	18	2
H^M 3 per-cent + 5s. + $\frac{1}{5}$th	1	17	3	2	7	4	3	3	1	4	10	0	6	18	7

office premiums, loaded for a uniform reversionary bonus (which is the bonus system of the office), but this is entirely a fortuitous circumstance. The view that Life Assurance premiums are fixed by competition alone, and that as long as the premium is sufficient for the risk and the expenditure, any loading for profit may be quite arbitrary, is, I believe, now almost non-existent; doubtless valuation by assumed low rates of interest and the prevalence of the systems of reversionary bonuses, calculated as a percentage on the sum assured, or on the sum assured and bonuses, have dispelled an idea, which was perfectly legitimate in the days when many offices valued by a true Table of Mortality and Interest,

and distributed cash bonuses in proportion to the surplus margins of the premiums, after providing for mortality risk and expenses. Scales of premiums are, it is true, often constructed limited in the minimum direction by the lowest premiums which the prevailing average rates of interest and mortality would permit, and in the maximum direction by the consideration of competitive premiums, which is a perfectly legitimate course, if sound actuarial principles as to loading be at the same time adhered to, but the loading for profit should be consistent with the bonus system, even if the rate of bonus provided for in the premium is not so great as in the premiums of some other companies having the same system, for if the profit loading be not calculated, there will most probably be unfairness as between entrants at different ages. Doubtless the effect of competition has been the main reason for the somewhat slow progress made with the adoption of the "Select" Table as a basis, and an equitable system of loading, coupled with the general disinclination to start new series of policyholders.

Until the new Experience appears we must, in my opinion, consider the existing "Select" Tables as our mortality basis, but with a view to forming some opinion as to the probability of a change being required in the interest basis for calculating office premiums, I have made some investigation into the matter of the present average rates of interest realized by the offices having this system of bonus, with the following results.

I have calculated by Mr. G. F. Hardy's formula for the effective rate of interest (deducting tax).

$$\text{Effective rate} = \frac{2\text{I}}{\text{A}+\text{B}-\text{I}}$$

(Where I is the interest item, less tax, in the Revenue account, A the fund at the beginning of the year, and B the fund at the end of the year), the average rates of interest of the 33 offices in the years 1875, 1885, and 1894.

Of the 18 uniform bonus offices only one was not in existence in 1875, and one other is the result of an amalgamation of two offices then having a separate existence, so that my figures for that year relate to 18 offices in 1875 as well as in 1885 and 1894. The average rate of these offices in 1875 was £4. 10*s*. 11*d*., in 1885 £4. 3*s*. 4*d*., and in 1894 £3. 19*s*. 5*d*.

In the Compound bonus class the average rate of interest of 14 offices in 1875 was £4. 12*s*. 8*d*., of the same number of

offices in 1885, £4. 6*s*. 4*d*., and of 15 offices in 1894 exactly 4 per-cent. It is a satisfactory circumstance that the offices in this class have, on the average, kept up a better rate of interest than in uniform bonus offices, as their system of bonus requires a larger margin of interest profit. These figures are merely of value for purposes of comparison, and they certainly indicate a steady fall in the rate of interest to the end of 1894, but I am inclined to put the average rate of interest in 1894 rather higher than 4 per-cent—say, at £4. 1*s*. per-cent, owing to the fact that a few offices earn interest from Reversions purchased which does not appear in the accounts, and the amount of such reversions is not excluded from the funds in applying the formula.

If, therefore, in 1881, when Dr. Sprague's suggested office premiums were published, the offices were making on an average nearly $4\frac{1}{4}$ per-cent, and the rate of interest adopted in calculating standard office premiums was 4 per-cent, should not the rate of interest to be used in premium calculations by many offices now, and by others in the near future, be $3\frac{3}{4}$ per-cent?

I think this is a question which cannot now be answered with regard to *all* of the offices, on the mere assumption of the average rate of interest earned by all; some offices have enlarged their powers of investment, and by cultivating reversionary transactions or colonial investments, or the debentures of first-class industrial companies, have succeeded in maintaining their average rate of interest well over 4 per-cent; in 1894, of the 18 uniform bonus offices eight realized an average rate of over 4 per-cent and 10 below that rate, the highest rate being £4. 3*s*. 6*d*. per-cent and the lowest £3. 15*s*. 5*d*. per-cent; of the 15 compound offices eight realized over 4 per-cent and seven under that rate, the highest rate being £4. 5*s*. 11*d*. per-cent and the lowest £3. 12*s*. 8*d*. It appears to me that offices which by energetic and sound finance are realizing, and are likely to realize for many years, a steady rate of over 4 per-cent may well hesitate to adopt a higher scale of premiums on a $3\frac{3}{4}$ per-cent basis for new entrants until they can foresee the impossibility of maintaining a 4 per-cent rate: with regard to offices which have already realized a lower rate, and appear to have settled down to a permanent lower rate, it only appears equitable that new entrants should be charged office premiums on a $3\frac{3}{4}$ per-cent basis, especially in the case of offices having large and safe reserves accumulated out of the premiums of those already on the books. In that case, assuming Sprague's Select Mortality basis

and $3\frac{3}{4}$ per-cent interest, loading for a 1 per-cent bonus and for expenses &c., by Sprague's method the office premiums would be approximately as follows:

	Age at Entry														
	20			30			40			50			60		
	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
Uniform Reversionary Bonus	2	2	1	2	10	3	3	5	9	4	11	4	6	16	4
Compound Reversionary Bonus	2	3	8	2	11	11	3	7	3	4	12	7	6	17	4

So that uniform bonus offices would in such case have to raise their rates to much the same scale as compound bonus offices should now charge on the existing 4 per-cent "standard" basis adopted in this paper, and compound bonus offices charging the present 4 per-cent standard scale would have to raise their premiums from ages 20 to 60 from 1*s.* 6*d.* to 1*s.* 11*d.* per £100. There is nothing very alarming in this, but looking at Tables III and IV, and observing how much lee-way has yet to be made up in the scales of premiums of many offices to reach such a standard, it would appear to me that offices looking forward to a permanently reduced rate of interest should recognize the fact of a reduced bonus-earning power, and adopt new scales of premiums for new entrants on a $3\frac{3}{4}$ per-cent basis and a "Select" mortality table, but loaded for a rather less rate of bonus than one per-cent per annum, say for three-fourths of one per-cent per annum, and for expenses and commission, &c., by Sprague's method. In this way many offices would be using a basis for their office premiums and a system of loading more in accordance with the actual facts of the case than their present scale, and more equitable in the matter of bonus as regards entrants at different ages. I am not at all an advocate for a dead level of premiums for all the offices, but I do think that a little more consideration should be given to the equitable calculation of the scales of premium in use in regard to mortality basis, interest basis, and loading.

In this connection it may be of interest to the members of the Institute if I here interpolate a few remarks as to the new premium tariff of the French Life Assurance Companies, which came into force on the 1 January 1894. I am indebted to M. Léon Marie, Fellow of the Institute of French Actuaries and Secretary of that Institute, for a copy of his pamphlet on the

subject, being an article which appeared in the 1895 Annual of the Society of Directors of Life Assurance Companies at Amsterdam. I learn from this that in France the custom has always prevailed of adopting a uniform tariff of life assurance premiums for all the companies, and legal decisions have given almost the force of law to this custom. For many years life assurance premiums were based on Duvillard's Table of Mortality and 4 per-cent interest, without loading, the exaggerated mortality rates shown by this falsely constructed table and the excess of the real rate of interest over the assumed rate of 4 per-cent being considered to be sufficient to provide the loading just in the same way as the pure Northampton 3 per-cent premiums were used for many years in this country without loading. At the Paris Exhibition of 1888 a committee of the four oldest French companies (The Compagnie d'Assurances Générales, Union, Nationale and Phénix) exhibited two tables. The one called R.F. represented the mortality of the French annuitants during the period 1819-1878. The other, called table A.F., represented the mortality of assured lives from 1819-1888. They were then adjusted by Mr. Woolhouse's method. Since the year 1888 the tables have been revised, the table R.F. (Rentiers Francais) has been completed by observations made between 1878-1890, and they have both been graduated by Makeham's method. On this table of mortality A.F. the premium tariffs are based, the rate of interest used being $3\frac{1}{2}$ per-cent, and the method of loading without-profit premiums being precisely similar to that of Dr. Sprague in his "Select" office premiums, except that the constant for expenses is much higher and the percentage loading rather lower. The reason given by M. Léon Marie for lowering the rate of interest to $3\frac{1}{2}$ per-cent is as follows:—"French 3 per-cents actually exceed par, and the $3\frac{1}{2}$ per-cents are a still less profitable investment, if one takes into consideration the conversion almost certain to take place eight years hence (from November 1894). The issues of the large railway companies with State guarantee do not produce a much better revenue. Real property and first-class mortgages bring in about 4 per-cent. Therefore the average return from safe investments is but little more than $3\frac{1}{2}$ per-cent." The committee has produced tariffs for all kinds of life assurance benefits on this basis, and each class of assurance comprises three tariffs.

The pure premium tariff (Tarif pur.).

The "stock-taking" premium tariff (Tarif d'inventaire); and The office premium tariff (Tarif commercial).

The valuation or "stock-taking" premium consists of the pure premium with the constant for expenses of management added, and the office premiums are derived from the valuation premiums by the introduction of the commission (procuration fee of £1 per-cent on the sum assured) and the expenditure for collection (as a percentage loading) into the formula, thus

if ϖ be the pure premium

ρ the constant for expenses

$+\rho=$valuation premiums (primes d'inventaire)

K the percentage loading for expenses of collection.

$$\text{Office premium per unit}=\left(\varpi+\rho+\frac{\cdot01}{1+a}\right)(1\cdot0\text{K}).$$

The resulting whole-life premium scale without profits is higher than that of the old French tariff. They are also higher than is usual in this country, and this may, I think, be accounted for by the fact that (1) over age 35 the mortality amongst French assured lives demands a higher premium than the English standard table; (2), that the rate of interest basis is $3\frac{1}{2}$ per-cent instead of the existing standard of 4 per-cent in this country; (3), that the constant used in the loading for expenses of management is very much higher than that used by Mr. Sprague, much more than counteracting the effect of a rather lower percentage loading.

The following are the resulting office without-profit premiums taken from the prospectus of one of the leading French Companies:

	Age 20			Age 30			Age 40			Age 50			Age 60		
	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
French, without-profit, Tariff Premium per £100 (1894)	1	18	3	2	8	0	3	4	7	4	12	10	7	3	4

The pure premiums by the A.F. and H^{M} and "Select" tables at $3\frac{1}{2}$ per-cent, and by the latter table at 4 per-cent (being the basis of Sprague's "Select" office premiums) are as follows:

Pure Premiums per £100.

	Age 20	Age 30	Age 40	Age 50	Age 60
A. F. $3\frac{1}{2}$ per-cent . . .	1·327	1·773	2·531	3·824	6·132
H^{M} $3\frac{1}{2}$ per-cent . . .	1·330	1·769	2·465	3·667	5·848
Select $3\frac{1}{2}$ per-cent . . .	1·471	1·814	2·477	3·618	5·681
Select 4 per-cent . . .	1·391	1·714	2·361	3·488	5·541

At the early assurance ages (20 to 35) the A.F. $3\frac{1}{2}$ per-cent pure premiums run very closely to the H^{M} $3\frac{1}{2}$ per-cent pure premiums.

The French Actuaries have evidently, in framing their premium tariff, faced the question of the fall in the rate of interest, the higher expenses at which business is now carried on, and the rate of mortality amongst assured lives, boldly and conscientiously, and although they do not appear to have adopted "Select" mortality tables, they have used the method of loading for expenses, &c., advocated by Dr. Sprague 14 years ago; the French Companies have, however, the advantage of their custom of all offices charging the same tariff of premiums, but notwithstanding this, I believe there has been a falling off of new Life Assurance business since the introduction of the higher tariff. In this country the recent tendency has been all in the direction of low rates of premium, and I fear the public would not be encouraged to assure if they thought the offices combined to charge what they would consider high rates of premium, however equitably they may have been constructed with regard to the circumstances of the business. The prevalent idea seems to be to effect the desired end of a low premium by "credit systems" and "discounted bonus" premiums, and inasmuch as this is accompanied by the maintenance in the prospectus of the full premium tables, I think this may tend towards even a greater readiness to put these full premium tables on an adequate basis, and so attain both ends, namely, the desire of the actuary to charge adequate premiums, and the desire of the public to be assured at a low premium.

I do not think it at all certain, however, that the fact of an office (apart from any combination of offices) charging adequate profit premiums to new entrants, even if they are rather higher than heretofore, would militate seriously against the extension of its business, as I find that offices charging adequate premiums for the maintenance of a satisfactory bonus are by no means left behind in the race for business. Indeed, I think that agents are more assisted and satisfied with the prospect of bonus maintenance or increase than with a low profit scale of premiums.

The French "primes d'inventaires", or stock-taking premiums, are interesting; they are the valuation premiums valued by pure annuities, and are formed by adding the constant for expenses of management (used in the loading of the premiums) to the pure premiums. The single premiums used in the valuation are

merely a commutation of the "primes d'inventaires." M. Léon Marie says of this method of valuation in his pamphlet:

"In the case in which the policy is by uniform annual "premiums throughout its duration, the reserve obtained by "this method is the same as the pure premium reserve. When "the number of premiums is less than the number of years in "the contract the reserve exceeds the pure premium reserve by "the difference between the actual present value of the expenses "of management remaining to be paid and that of the loadings to "be received for this purpose."

I believe the usual French system of bonus distribution is a uniform percentage each year on the total premiums paid on the policy, so that it has no reference to the system of bonus under consideration, but the with-profit premiums for contracts lasting at least 10 years, are by the tariff formed by multiplying the without-profit premiums by $\frac{10}{9}$.

Before concluding the subject of the rates of premiums to be charged, I give here specimens of bonus-loading I have calculated in the course of writing this paper, which I think illustrate several curious points in connection with them.

Bonus-Loading for One per-cent per annum.

Age at Entry	Sprague's "Select" 4 Per-cent	HM 4 Per-cent	Sprague's "Select" 3¾ Per-cent	HM 3¾ Per-cent	Sprague's "Select" 3½ Per-cent	HM 3½ Per-cent
	Uniform Reversionary Bonus					
20	·364	·356	·390	·378	·416	·404
30	·427	·422	·450	·445	·473	·469
40	·501	·497	·523	·519	·545	·541
50	·579	·576	·597	·595	·618	·615
60	·645	·642	·660	·658	·678	·675
	Compound Reversionary Bonus					
20	·437	·433	·468	·462	·500	·491
30	·499	·496	·526	·523	·553	·551
40	·567	·565	·592	·590	·617	·615
50	·635	·636	·656	·656	·677	·676
60	·688	·685	·705	·703	·723	·721

It will be observed what a very little difference the Table of Mortality used makes in the calculated loadings. I think the cause of this is that the Mortality Table showing a light rate

of mortality involves in the bonus-loading the equivalent of the bonuses realized by a long liver, which is a set off against the increase of premium arising from the use of a Mortality Table showing a higher rate of mortality; as evidence of this the differences between bonus-loadings by the "Select" and the H^M Tables at the same rate of interest are, although small in both cases, less in the "Compound" Reversionary Class than the "Uniform" Reversionary Class.

I find that on a 3 per-cent basis the H^M Table and Morgan's Equitable Table give identical bonus-loadings. Another feature of these loadings is that for different rates of interest, but for the same mortality table and age, the simple bonus-loadings and compound bonus-loadings are approximately proportionate: for example, in the case of the H^M Table, having calculated the H^M 4 per-cent compound bonus-loading at age 20 to find the H^M $3\frac{1}{2}$ per-cent compound bonus-loading at the same age, given the corresponding simple bonus-loadings (which are more easily calculated).

$$\cdot356 : \cdot404 :: \cdot433 : \frac{\cdot433 \times \cdot404}{\cdot356} = \cdot491$$

$$= 3\tfrac{1}{2} \text{ per-cent compound bonus-loading.}$$

To show that the loading for the annual interim bonus is small compared with that for the quinquennial bonus, I submit the following figures:

Loading for an Interim Bonus of 1 *per-cent per annum.*

Age at Entry	Uniform Interim Bonus		Compound Interim Bonus	
	"Select" 4 per-cent	H^M 4 per-cent	"Select" 4 per-cent	H^M 4 per-cent
20	·035	·032	·048	·046
30	·045	·043	·059	·057
40	·062	·061	·077	·077
50	·089	·089	·104	·104
60	·126	·126	·140	·140

For a $3\frac{3}{4}$ per-cent basis the loadings for interim bonus would not differ more than about a penny from these 4 per-cent values.

If an intermediate bonus of 1 per-cent per annum were allotted for each year entered upon between the date of assurance and the first quinquennial division, the cost of this in the rate of premium charged would be very slight—from 1*d.* at age 20 to about 10*d.* at age 60.

In close connection, however, with the charging of suitably constructed rates of premium by an office under this system of bonus, is the method and basis of valuation adopted. Of the method I have little to say, as the pure premium method is now so generally adopted in this country: of the 18 uniform bonus offices, three only valued on the last occasion by a gross premium method, and of the 15 compound bonus offices, one only. But the basis is important, as a considerable margin between the average rate of interest earned, and the valuation rate is essential for the maintenance of even the minimum rate of bonus provided by the loading of the premiums charged. The cash value of a uniform or compound reversionary bonus increases at each valuation, and the bonus-loading of a premium remains constant throughout the duration of the policy, but if the valuations be made by a true rate of interest, and true table of mortality, the same cash surplus would be derived from the premiums at each distribution, resulting when converted into reversion in a decreasing reversionary bonus. Assuming a "true" valuation, therefore, it is necessary to divide only a portion of the available surplus and retain the remainder as a bonus reserve to accumulate and provide for the uniform reversionary bonus.

Dr. Sprague, in his paper in 1857 (*J.I.A.*, vii, 61), suggested the following plan in order to ascertain whether at any valuation the minimum rate of bonus provided for by the premiums may be declared.

First calculate the cash surplus required for a bonus for the five years at the rate of bonus assumed in the bonus loading and call the balance of the surplus . R.

Let the present value of future additions at the rate assumed in the premiums . . . . . . =B.

Let the present value of the future bonus loadings =L.

Then if B be $>L+R$ the rate of bonus from the premiums must not be as high as the assumed rate,

and if B be $<L+R$ at least the assumed rate may be declared.

But valuations by true rates of interest and mortality coupled with estimates of bonus reserves by this method are now unusual, although for testing valuation results and bonus reserves they might be desirable. It has been ascertained by the investigations of various actuaries that (apart from the profit from annual surplus margins and other sources) valuation by a rate of interest about one per-cent less than that realized will have the desired effect of producing increasing cash surpluses at each

quinquennial division, which will, in the case of assurances by single premiums, approximately produce (when converted into reversionary values) uniform reversionary bonuses, but in the case of assurances by annual premiums a margin of interest of as much as one per-cent would produce increasing reversionary bonuses at each quinquennial division.

I give here, by way of illustration, the quinquennial reversionary bonuses corresponding to Mr. Meikle's results for a one per-cent margin of interest (*J.I.A.*, xi, 251) and Carlisle 3 per-cent single and annual premiums for age at entry 30, and Mr. Sunderland's similar results (*J.I.A.*, xxvi., 370) for H^M 3 per-cent annual premiums, all loading spent, the mortality experienced being the same as that assumed in the valuation.

Quinquennial Reversionary Bonuses per £100 resulting from an Interest Margin of 1 per-cent per annum.

Age	Age at Entry 30		
	(Meikle) Carlisle 3 per-cent Single Premium	(Meikle) Carlisle 3 per-cent Annual Premium	(Sunderland) H^M 3 per-cent Annual Premium
35	5·32	·51	·53
40	5·32	1·16	1·24
45	5·40	1·78	1·94
50	5·36	2·31	2·63
55	5·37	2·89	3·28
60	5·53	3·54	3·92
65	5·85	4·19	4·54
70	6·25	4·88	5·12
75	6·69	5·59	5·70
80	7·66	6·68	6·26

By reference to Table B, Experience H^M 4 per-cent, of Mr. Sunderland's paper, it will be observed that an annual surplus margin will, with its own interest profit (assuming a 3 per-cent valuation), produce approximately a uniform reversionary bonus for each age at entry, but not the same uniform reversionary bonus for all ages at entry: if the same reversionary bonus for all ages at entry is to be given, an annual surplus margin increasing rapidly with the age at entry must exist, and adding the interest profit from reserves an increasing reversionary bonus would of course result.

Mr. C. J. Harvey in his paper on a formula for returning to the assured their contributions to the surplus (*J.I.A.*, xxiv, 173), brought out the following results, assuming H^M

mortality, $4\frac{1}{2}$ per-cent interest realized and 4 per-cent used in the valuation.

Reversionary Bonuses per £1000 *assured.*

Age at Entry	Assumed Office Annual Premium	Annual Surplus Margin	Years in Force							
			5	10	15	20	25	30	35	40
20	1·938	·396	84	80	78	76	74	72	72	71
30	2·467	·474	80	78	75	74	73	72	72	73
40	3·263	·548	74	72	71	71	71	72	73	75
50	4·588	·617	66	66	67	69	70	73	75	77
60	7·092	·822	71	73	76	79	82	85	86	88

In the discussion on Mr. Sunderland's paper (*J.I.A.*, xxvi, 383) Mr. Harvey stated with reference to his own paper, that he there showed "that where there was a small surplus interest of say $\frac{1}{2}$ per-"cent, a reversionary bonus of a constant percentage on the sums "assured for each premium paid during a quinquennium, gave "results identical with those of his formula, and where the surplus "interest was a little higher, the allocation of a reversionary bonus "by a percentage on the sum assured and previous bonuses gave "results very near the truth indeed."

The problem, therefore, so far as profit from premiums and interest is concerned, is to find such a scale of premiums as will provide annual surplus margins at each age which, in conjunction with a practical margin of interest-profit, will produce uniform reversionary bonuses. The complicated conditions of such a problem, and the fact that the margins of interest profit in practice are usually more than one-half per-cent are sufficient to indicate the extremely artificial and cramped nature of such a system of bonus.

As to the interest-margin now existing in practice.

Of the uniform bonus offices valuing by the pure premium method,

5 value on the H^M and $H^{M(5)}$ 3 per-cent basis and three of these could, in my opinion, having regard to the amount of undivided surplus carried forward, go to a $2\frac{1}{2}$ per-cent valuation, their interest margins ranging from about $\frac{3}{4}$ to 1 per-cent, assuming a 3 per-cent basis.

3 value by the H^M 3 per-cent Table, and of these probably one could go to a $2\frac{1}{2}$ per-cent valuation, their interest margins ranging from about 16*s.* to one guinea, assuming a 3 per-cent basis.

1 values on the H^M and $H^{M(5)}$ $3\frac{1}{2}$ per-cent basis, but could go to an H^M 3 per-cent valuation, with an interest margin of about one per-cent on a 3 per-cent basis.

2 value on the H^M $3\frac{1}{2}$ per-cent basis, but could go to a 3 per-cent valuation, their interest margins, assuming a 3 per-cent basis, being very nearly one per-cent in each case.

1 values on an H^M $3\frac{1}{4}$ per-cent basis, with an interest margin of about 18*s.* 6*d.*

3 value on an H^M $3\frac{1}{2}$ per-cent basis, one with a margin of interest of about 5*s.* 5*d.*, another of about 7*s.* 10*d.*, and a third of about 13*s.* 6*d.*

Of the Compound bonus offices valuing by the pure premium method,

1 values on the H^M and $H^{M(5)}$ $2\frac{1}{2}$ per-cent basis, with a margin of interest of about $1\frac{1}{2}$ per-cent.

5 value on the H^M and $H^{M(5)}$ 3 per-cent basis, with margins of interest ranging from about 18*s.* to 25*s.* on a 3 per-cent basis, of which one could, in my opinion, go to a $2\frac{3}{4}$ per-cent valuation.

2 value on the H^M 3 per-cent basis, with margins of interest of about £1. 3*s.* 3*d.* and £1. 5*s.* 11*d.*

3 value on the H^M $3\frac{1}{2}$ per-cent basis, but could go to a 3 per-cent valuation with margins of interest on a 3 per-cent basis of about 12*s.* 8*d.*, £1. 0*s.* 11*d.* and £1. 2*s.* 1*d.*

1 values on the H^M $3\frac{1}{4}$ per-cent basis, with an interest margin of about 17*s.* 9*d.*

1 values on the H^M $3\frac{1}{2}$ per-cent basis, with an interest margin of about 8*s.* 2*d.*

1 values on the H^M 4 per-cent basis, carrying forward a special reserve to attain a $3\frac{1}{2}$ per-cent valuation, the interest margin on a $3\frac{1}{2}$ per-cent basis being about 9*s.* 3*d.*

If reference be made to Table VI appended to this paper, it will be observed that, assuming Sunderland's hypothetical annual surplus margins with a valuation by the H^M 3 per-cent

Table, a margin of interest of between one-half per-cent and three-quarters per-cent is sufficient for a uniform reversionary bonus and of between three-quarters per-cent and one per-cent for a compound reversionary bonus, if H^M mortality be experienced. If, therefore, Sunderland's hypothetical premiums may be considered suitable and fair average premiums, many offices having premiums approximate thereto have adequate, and, in some cases, more than adequate interest margins to maintain their bonus systems, this eminently satisfactory state of things having resulted from the foresight of actuaries in gradually strengthening the reserves of offices during the last 20 years, and I think this strengthening of reserves has been brought about conjointly by the necessities of this system of bonus and by the probable fall in the rate of interest, and not by the latter influence only.

As Mr. Sunderland's hypothetical average premiums are referred to, it is desirable to reproduce them here.

Age 20	£1	19	0
„ 30	2	9	0
„ 40	3	5	0
„ 50	4	12	0
„ 60	7	0	0

From these hypothetical premiums (which will be observed are practically the same as the H^M 4 per-cent office premiums loaded for a compound bonus by Sprague's method), by deducting 5 per-cent and 3*s*., and the valuation premium (H^M 3 per-cent), are derived certain annual surplus margins.

The constant for expenditure to be deducted (3*s*. in this case), of course, varies in different offices, and I have come to the conclusion that the following is a fairly satisfactory method of arriving at such a constant for any particular office.

Taking the consolidated revenue account of the fifth schedule and adding together the expenses and commission items and deducting therefrom (1) 5 per-cent of the premiums, (2) one per-cent of the sums assured in respect of the total new business of the quinquennium, less re-assurances, (3) one per-cent on annuity considerations paid in the five years, (4) a small sum for each annuity contract in force at the close of the quinquennium, I divide the remainder by the sums assured in force at the close of the quinquennium, less re-assurances, (divided by 100), and divide the result by five. It is surprising how satisfactorily the results accord with what one would expect to be the relative strain of expenditure of the various offices. I have calculated these

constants in several cases, and I find that, in the case of a progressive office, which might fairly be considered to be most economically managed, the constant works out at precisely 3*s*., and in another progressive office, with a somewhat high ratio of expenses and commission to premium income, at nearly 6*s*., so that I am inclined to consider these figures as the limits of value of such constants at the present time in offices not extravagantly managed. The method also has the advantage of according with the assumptions made in loading the "standard" office premiums assumed in this paper. Having calculated this constant for any particular office, it is possible to derive from the office premiums of that office, by the method above explained, the annual surplus margins at each age at entry, and thence, with the aid of Mr. Sunderland's Tables, derive the resulting bonuses.

In order to illustrate the effect of annual surplus margins, combined with interest profit in producing bonuses, and also the effect on these bonuses of the initial commission of £1 per £100 assured, I have, with the aid of Mr. Sunderland's A, B and C and A′, B′ and C′ Tables (*J.I.A.*, xxvi, 368–378), constructed Tables VI, VII and VIII, appended to this paper. Throughout these tables H^M 3 per-cent valuations are assumed.

Table VI relates to reversionary bonuses arising from annual surplus margins derived from Sunderland's hypothetical average premiums, combined with various realized rates of interest and assuming H^M mortality experience. In the last column I have placed the bonuses resulting from Harvey's hypothetical average premiums under the general assumptions of Table VI and assuming $3\frac{1}{2}$ per-cent interest realized.

Table VII exhibits reversionary bonuses arising from the annual surplus margins of the "Select" 4 per-cent office premiums, assumed as a standard in this paper, or the H^M 4 per-cent office premiums similarly loaded, combined with various rates of interest, and assuming H^M mortality experience.

Table VIII shows (in all but the last two columns) the effect on bonuses of experiencing the "Select" mortality, with 4 per-cent interest, in combination with Sunderland's average office premiums, H^M 4 per-cent office premiums, or "Select" 4 per-cent office premiums. I think a very fair idea of the effect of experiencing "Select" mortality, instead of H^M mortality, can be gathered by comparison of the results in this table, with the corresponding 4 per-cent results in Tables VI and VII.

In the last two columns of Table VIII I have placed the

reversionary bonuses, resulting from annual surplus margins, as in the foot note to the Table (being 50 per-cent of the difference between the H^M 3 per-cent pure premiums for the several ages at entry, and the Clergy Mutual Experience* 3 per-cent pure premiums). These columns may be useful in forming an idea of the resulting reversionary bonuses, derived by dividing miscellaneous profits (including mortality profit) in proportion to such annual margins. In examining the tables, it must be borne in mind that it is assumed that the minimum expected bonuses under with-profit office premiums are usually 1 per-cent per annum, and these minimum expected bonuses are placed in columns (2) and (3) for comparison. It will be observed throughout these tables that the bonuses, at the end of the first five years, are considerably affected by the £1 per £100 initial commission, and the bonuses of the early years of assurance are also affected by the putting aside of strong reserves at a low rate of interest. Subject to these generalizations, I draw the following conclusions from these tables.

Table VI is not altogether satisfactory for purposes of comparison of uniform bonus offices, inasmuch as Sunderland's hypothetical premiums are almost the same as H^M 4 per-cent office premiums, when loaded for a compound bonus; but it is useful, in showing at a glance, under similar conditions, the effect of various realized rates of interest.

With regard to uniform bonus offices having premiums equal to, or approximating to those based on the H^M Table (see Table VII), and valuing by that table, those offices experiencing H^M mortality and realizing $3\frac{3}{4}$ per-cent interest (and in a greater degree 4 per-cent interest), are really earning increasing reversionary bonuses, and not uniform bonuses, and by giving uniform bonuses they are favouring the policies of the younger entrants and policies of comparatively short duration at the expense of the older entrants and policies of long duration.

If $3\frac{1}{2}$ per-cent only be realized the bonuses are, of course, reduced, but they are more of the nature of a uniform bonus, and the inequity of the $3\frac{3}{4}$ per-cent results is modified.

In the case of uniform bonus offices having premiums on the "Select" standard and experiencing H^M mortality, the realization of $3\frac{3}{4}$ per-cent interest practically produces bonuses of the compound form (and 4 per-cent interest would result in bonuses increasing more rapidly than the usual quinquennial compound

* The pure premiums at age 20 being approximated.

bonus). The realization of only $3\frac{1}{2}$ per-cent interest would produce better results than when the premiums are based on the H^M Table, nearer to those of the minimum expected bonuses at the younger ages at entry, and with the tendency of the $3\frac{3}{4}$ per-cent results to favour policies of comparatively short duration modified; but there would be a tendency to give to policies taken out at the older ages and of comparatively short duration too much bonus.

With regard to the compound bonus system, assuming H^M mortality to be experienced, a realized rate of interest of $3\frac{3}{4}$ per-cent combined with Select 4 per-cent office premiums (loaded for a compound bonus) seems to me to give the most satisfactory general results, but premiums on a similar basis with a simple bonus-loading seem also to give bonuses nearly equal to the minimum expected (illustrating the view that loading for a uniform bonus is nearly sufficient with the interest profit on bonus reserves to effect the compounding of the bonus).

By comparison of the results in Table VIII with the corresponding results in Table VI and VII, it appears to me that the general effect upon the bonuses of offices experiencing "Select" mortality instead of "H^M", is to arrest their increase, having a greater effect on the policies of young entrants and policies of short duration than on policies of older entrants and long duration, its effect decreasing as the age at entry and the duration of the policy increases. The bonuses arising from the surplus annual margins are much the same in both experiences as we should expect, but the bonuses arising from interest profit are affected in the way stated.

Mr. Sunderland (Vol. xxvi, 367) says: "for Dr. Sprague's "Select Mortality Tables the bonuses are, generally speaking, "smaller than those for the H^M Experience, except for ages at "entry 50 and 60; for age at entry 20 the differences are very "considerable, and in fact for the former experience no bonus "arises until the third quinquennium." Some irregularities of the results would, I presume, disappear if the valuations were made by H^M and $H^{M(5)}$ Tables instead of H^M Tables, but I think the general results show that an office experiencing "Select" mortality would have to realize a rather higher rate of interest in order to maintain its bonus system, than an office experiencing only H^M mortality.

The bonus system, therefore, as regards interest-profit, is well maintained if $3\frac{3}{4}$ per-cent be realized in the case of simple bonus offices, and 4 per-cent in the case of compound bonus

offices, making 3 per-cent valuations: and even should the margin of interest be slightly below this, provided H^M mortality be experienced; although the bonuses would, of course be less, there would be less inequity as regards the distribution of the profits amongst policies for various ages at entry and durations.

With regard to the balance of profit and loss from mortality experience, realization of investments and non-participating business, and the profits from surrenders and lapses, there is no option in this extremely rigid system of bonus, but to distribute it in the same form as the other surplus. In the "contribution" method of distribution such profit is sometimes distributed in proportion to the premiums paid or to the annual surplus margins, and the last two columns in Table VIII show that in the latter case (for the annual surplus margins assumed) an equal reversionary bonus would practically result.

Table IX has been added to the above-mentioned tables in order to show approximately, according to the conditions of actual practice, the combined effect of various rates of interest, scales of premium, and *rates of expenditure*, the constants of expenditure in the previous tables having been assumed throughout at 3*s.* only per £100 assured. The differences between the bonuses resulting from annual surplus margins and surplus interest and a 30*s.* simple or compound bonus, as the case may be, are placed in juxtaposition, in order to show the incidence of the portion of the bonus resulting from miscellaneous-profit (including mortality-profit) at various ages and durations. A bonus at the rate of 30*s.* is assumed as an average rate for purposes of comparison.

As to the share of profits allotted to policyholders in proprietary offices, the majority of such offices now give as much as 90 per-cent of the divisible surpluses to the participating policyholders. Owing to the more liberal spirit of boards of directors and shareholders, there has been a gradual tendency (growing with the strengthening of office reserves) towards this percentage becoming more general for some years past, and any less percentage would seem to be undesirable from the point of view of bonus prospects unless the shareholders are the means of influencing new business sufficiently profitable to compensate for their too ample share of the profits.

In conclusion, the general results of my investigation are as follows:—

(1) That existing scales of premiums in many offices having this system of bonus require considerable revision in order to render their bonus distributions equitable as between entrants at different ages.

(2) That under this bonus system there is a general tendency to give too much bonus to policies of short duration on the books.

(3) That, in the case of uniform bonus offices, at least a margin of interest profit of three-quarters per-cent is required when "Select" mortality is experienced, and at least a margin of one per-cent in the case of the compound offices under similar circumstances.

(4) That the fall in the rate of interest has not yet been sufficient to render it necessary to reduce the valuation rate of interest in all cases below 3 per-cent, inasmuch as the average rate of interest now realized by many offices gives a sufficient margin of interest-profit for the exigencies of this bonus system as regards equitable distribution.

(5) That having regard to the various tendencies to diminish the bonuses of offices, it is desirable to decline to give estimates of future bonuses based on past results, and also to limit the percentage of bonuses discounted for reduction of premiums to the minimum rate which under present circumstances may be considered permanent.

(6) That this system of bonus distribution, and especially the "uniform bonus" form of it, although having the advantage of simplicity, is of too rigid a nature to be suitable to the varying circumstances of life offices, if we regard as important the equitable distribution of profit amongst those who have contributed to it. It lacks the elasticity of the "contribution" method, but in offices experiencing average results, the compound form of it would give bonuses approximately the same as the contribution method, so that, in such cases, the merit of simplicity may be combined with the principles of equity.

TABLE VI.

Reversionary Bonuses resulting at the end of each quinquennium from the date of Assurance for each £100 Assured, assuming Hypothetical Scales of Premium and various Realized Rates of Interest. Valuation H^M *3 per-cent.* H^M *Mortality Experience.*

Age	Uniform* Bonus of 1% per annum	Compound* Bonus of 1% per annum	Sunderland's Hypothetical Table, realized interest 3%	Sunderland's Hypothetical Table, realized interest 3¼%	Sunderland's Hypothetical Table, realized interest 3½%	Sunderland's Hypothetical Table, realized interest 3¾%	Sunderland's Hypothetical Table, realized interest 4%	Sunderland's Hypothetical Table, realized interest 4¼%	Harvey's Hypothetical Table, realized interest 3½%
	Age at Entry, 20.								
25	5	5	·96	1·08	1·18	1·29	1·39	1·50	1·00
30	5	5·25	3·84	4·15	4·46	4·77	5·10	5·42	4·29
35	5	5·51	3·51	4·01	4·51	5·03	5·57	6·14	4·35
40	5	5·79	3·20	3·89	4·57	5·31	6·07	6·86	4·43
45	5	6·08	2·91	3·78	4·64	5·57	6·56	7·61	4·50
50	5	6·38	2·66	3·69	4·72	5·87	7·10	8·41	4·58
55	5	6·70	2·44	3·63	4·81	6·16	7·62	9·22	4·68
60	5	7·04	2·24	3·57	4·90	6·46	8·17	10·06	4·78
65	5	7·39	2·08	3·56	5·04	6·79	8·76	10·97	4·92
70	5	7·76	1·94	3·55	5·16	7·11	9·35	11·89	5·05
75	5	8·14	1·83	3·58	5·33	7·47	9·99	12·89	5·19
80	5	8·56	1·75	3·61	5·47	7·84	10·64	13·93	5·36
85	5	8·98	1·69	3·67	5·63	8·23	11·34	15·05	5·52
	Age at Entry, 30.								
35	5	5	1·10	1·23	1·35	1·48	1·61	1·74	1·56
40	5	5·25	3·46	3·81	4·15	4·51	4·86	5·22	4·35
45	5	5·51	3·15	3·70	4·26	4·84	5·43	6·05	4·44
50	5	5·79	2·88	3·63	4·37	5·17	6·01	6·88	4·55
55	5	6·08	2·63	3·56	4·49	5·50	6·59	7·73	4·65
60	5	6·38	2·42	3·52	4·62	5·85	7·18	8·60	4·78
65	5	6·70	2·25	3·51	4·77	6·21	7·79	9·51	4·92
70	5	7·04	2·10	3·52	4·92	6·56	8·39	10·40	5·07
75	5	7·39	1·98	3·53	5·08	6·93	9·03	11·39	5·22
80	5	7·76	1·89	3·57	5·23	7·31	9·68	12·38	5·39
85	5	8·14	1·83	3·63	5·43	7·70	10·35	13·42	5·58
	Age at Entry, 40.								
45	5	5	1·45	1·61	1·75	1·90	2·06	2·21	1·90
50	5	5·25	3·37	3·76	4·16	4·58	4·99	5·42	4·30
55	5	5·51	3·08	3·71	4·33	4·99	5·66	6·37	4·46
60	5	5·79	2·83	3·68	4·51	5·42	6·35	7·33	4·64
65	5	6·08	2·63	3·67	4·70	5·84	7·03	8·30	4·81
70	5	6·38	2·45	3·66	4·87	6·23	7·68	9·25	4·98
75	5	6·70	2·31	3·69	5·06	6·64	8·35	10·24	5·17
80	5	7·04	2·21	3·73	5·25	7·04	9·03	11·23	5·36
85	5	7·39	2·14	3·80	5·46	7·44	9·70	12·26	5·57
	Age at Entry, 50.								
55	5	5	1·84	2·02	2·21	2·40	2·59	2·78	2·11
60	5	5·25	3·41	3·87	4·34	4·82	5·31	5·81	4·24
65	5	5·51	3·16	3·88	4·61	5·36	6·14	6·94	4·51
70	5	5·79	2·95	3·89	4·84	5·87	6·92	8·03	4·75
75	5	6·08	2·79	3·94	5·08	6·34	7·69	9·13	5·00
80	5	6·38	2·66	3·99	5·32	6·83	8·44	10·18	5·23
85	5	6·70	2·58	4·07	5·56	7·27	9·16	11·24	5·47
	Age at Entry, 60.								
65	5	5	2·28	2·51	2·75	3·00	3·24	3·46	3·40
70	5	5·25	3·63	4·18	4·73	5·28	5·84	6·43	5·35
75	5	5·51	3·42	4·25	5·08	5·95	6·85	7·78	5·69
80	5	5·79	3·27	4·34	5·41	6·56	7·75	9·01	6·01
85	5	6·08	3·16	4·44	5·71	7·11	8·60	10·20	6·31

Annual Surplus Margins.

	AGE AT ENTRY 20	30	40	50	60
Sunderland's Hypothetical Table ...	·275	·298	·348	·420	·513
Harvey's Hypothetical Table ...	·264	·314	·362	·408	·600

Constant of Expenditure . . 3/- per £100 assured.

N.B.—In Tables VI and VII the 3¾ and 4¼ per-cent results are interpolated by the formula of interpolation suggested by Mr. Sunderland (*J.I.A.*, xxvi, 363): the 3¼ per-cent results are merely the "mean" of the 3 per-cent and 3½ per-cent results.

Table VII.

Reversionary Bonuses resulting at the end of each quinquennium from the date of Assurance for each £100 Assured, assuming H^M 4 *per-cent, or "Select" 4 per-cent Office Premiums, and various realized rates of interest. Valuation* H^M 3 *per cent.* H^M *Mortality Experience.*

Age	Uniform Bonus of 1% per annum	Compound Bonus of 1% per annum	H^M 4% Office Premiums, Simple Bonus loading, Realized interest 3¼%	H^M 4% Office Premiums, Simple Bonus loading, Realized interest 3½%	H^M 4% Office Premiums, Simple Bonus loading, Realized interest 4%	H^M 4% Office Premiums, Compound Bonus loading, Realized interest 3½%	H^M 4% Office Premiums, Compound Bonus loading, Realized interest 4%	Select 4% Office Premiums, Simple Bonus loading, Realized interest 3¼%	Select 4% Office Premiums, Simple Bonus loading, Realized interest 3½%	Select 4% Office Premiums, Simple Bonus loading, Realized interest 4%	Select 4% Office Premiums, Compound Bonus loading, Realized interest 3½%	Select 4% Office Premiums, Compound Bonus loading, Interest 4%
	Age at Entry, 20.											
25	5	5	−·04	·07	·16	1·30	1·40	2·35	2·47	2·58	3·58	3·69
30	5	5·25	3·31	3·61	3·91	4·78	5·11	5·55	5·90	6·25	6·95	7·33
35	5	5·51	3·43	3·92	4·43	5·03	5·59	5·54	6·10	6·69	7·11	7·73
40	5	5·79	3·56	4·24	4·95	5·31	6·08	5·55	6·33	7·15	7·30	8·16
45	5	6·08	3·68	4·54	5·47	5·58	6·57	5·56	6·56	7·62	7·49	8·61
50	5	6·38	3·81	4·88	6·02	5·88	7·10	5·59	6·82	8·13	7·71	9·11
55	5	6·70	3·94	5·21	6·56	6·17	7·63	5·64	7·09	8·65	7·96	9·62
60	5	7·04	4·07	5·51	7·10	6·46	8·18	5·70	7·36	9·20	8·22	10·17
65	5	7·39	4·23	5·86	7·69	6·79	8·77	5·81	7·69	9·80	8·54	10·77
70	5	7·76	4·37	6·18	8·26	7·11	9·36	5·92	8·01	10·41	8·83	11·39
75	5	8·14	4·53	6·53	8·83	7·48	10·00	6·05	8·37	11·07	9·22	12·08
80	5	8·55	4·70	6·89	9·48	7·86	10·65	6·22	8·76	11·76	9·63	12·81
85	5	8·98	4·85	7·26	10·14	8·14	11·36	6·39	9·17	12·51	10·05	13·61
	Age at Entry, 30.											
35	5	5·	·53	·65	·78	1·63	1·76	1·19	1·32	1·45	2·25	2·38
40	5	5·25	3·38	3·72	4·06	4·65	5·00	4·00	4·36	4·71	5·24	5·60
45	5	5·51	3·54	4·08	4·66	4·97	5·57	4·12	4·69	5·28	5·53	6·15
50	5	5·79	3·69	4·46	5·26	5·30	6·15	4·24	5·04	5·86	5·84	6·72
55	5	6·08	3·84	4·81	5·85	5·63	6·72	4·36	5·37	6·44	6·15	7·27
60	5	6·38	4·00	5·18	6·45	5·98	7·31	4·50	5·73	7·04	6·48	7·86
65	5	6·70	4·17	5·56	7·06	6·33	7·92	4·65	6·08	7·65	6·82	8·49
70	5	7·04	4·34	5·91	7·65	6·69	8·52	4·81	6·45	8·25	7·19	9·08
75	5	7·39	4·51	6·28	8·28	7·06	9·17	4·97	6·81	8·89	7·55	9·73
80	5	7·76	4·69	6·66	8·91	7·44	9·83	5·14	7·20	9·54	7·87	10·41
85	5	8·14	4·87	7·02	9·54	7·82	10·50	5·32	7·56	10·20	8·32	11·10
	Age at Entry, 40.											
45	5	5·	1·12	1·27	1·42	1·98	2·14	1·28	1·43	1·58	2·11	2·27
50	5	5·25	3·57	3·98	4·38	4·66	5·07	3·72	4·13	4·53	4·78	5·19
55	5	5·51	3·78	4·41	5·07	5·06	5·74	3·91	4·55	5·22	5·18	5·86
60	5	5·79	3·99	4·86	5·77	5·49	6·43	4·12	5·00	5·92	5·60	6·55
65	5	6·08	4·19	5·30	6·45	5·91	7·10	4·32	5·44	6·60	6·02	7·22
70	5	6·38	4·38	5·70	7·10	6·30	7·75	4·50	5·88	7·25	6·41	7·87
75	5	6·70	4·58	6·10	7·77	6·70	8·43	4·69	6·24	7·92	6·81	8·55
80	5	7·04	4·77	6·50	8·43	7·11	9·11	4·89	6·74	8·58	7·22	9·23
85	5	7·39	4·98	6·90	9·08	7·51	9·78	5·10	7·04	9·24	7·63	9·91
	Age at Entry, 50.											
55	5	5·	1·62	1·83	2·01	2·37	2·56	1·17	1·36	1·54	1·86	2·05
60	5	5·25	3·80	4·27	4·75	4·79	5·28	3·35	3·82	4·29	4·31	4·78
65	5	5·51	4·09	4·83	5·59	5·33	6·11	3·68	4·39	5·13	4·86	5·62
70	5	5·79	4·34	5·34	6·38	5·83	6·87	3·94	4·91	5·93	5·38	6·41
75	5	6·08	4·60	5·83	7·14	6·32	7·66	4·21	5·40	6·69	5·86	7·17
80	5	6·38	4·84	6·29	7·87	6·78	8·41	4·45	5·87	7·41	6·32	7·91
85	5	6·70	5·08	6·75	8·59	7·25	9·13	4·70	6·32	8·11	6·79	8·62
	Age at Entry, 60.											
65	5	5·	2·06	2·31	2·55	2·64	2·89	·71	·94	1·16	1·30	1·53
70	5	5·25	4·08	4·63	5·19	4·95	5·52	2·77	3·29	3·82	3·64	4·18
75	5	5·51	4·48	5·31	6·19	5·62	6·52	3·18	3·99	4·82	4·34	5·18
80	5	5·79	4·80	5·91	7·08	6·22	7·41	3·54	4·59	5·70	4·94	6·06
85	5	6·08	5·10	6·45	7·92	6·77	8·26	3·84	5·13	6·49	5·48	6·86

Annual Surplus Margins.

		20	30	40	50	60
H^M 4% Office Premium, Simple Bonus	...	·197	·235	·290	·356	·427
H^M 4% Office Premium, Compound Bonus	...	·276	·310	·356	·416	·470
Select 4% Office Premium, Simple Bonus	...	·351	·286	·305	·304	·248
Select 4% Office Premium, Compound Bonus	..	·422	·357	·368	·360	·295

Constant of Expenditure . . 3/- per £100 assured.

Table VIII.

Reversionary Bonuses resulting at the end of each quinquennium from the date of Assurance for each £100 assured, assuming various Scales of Premiums. Valuation H^M 3 *per-cent. Sprague's "Select" Mortality Experience: in the last two columns* H^M *Experience.*

Age	Uniform Bonus of 1 % per annum	Compound Bonus of 1 % per annum	Sunderland's Hypothetical Table Select 4 % Experience	H^M 4 % Office Premium Simple Bonus Loading Select 4 % Experience.	H^M 4 % Office Premium Compound Bonus Loading Select 4 % Experience	Select 4 % Office Premium Simple Bonus Loading Select 4 % Experience	Select 4 % Office Premium Compound Bonus Loading Select 4 % Experience	Bonuses from Hypothetical Annual Surplus Margins as stated in footnote. Experience H^M 3½ %	Bonuses from Hypothetical Annual Surplus Margins as stated in foot-note. Experience H^M 4 %
	Age at Entry, 20.								
25	5	5	−2·05	−3·32	−2·04	− ·83	·32	2·75	2·77
30	5	5·25	1·10	− ·05	1·12	2·23	3·28	2·62	2·67
35	5	5·51	3·92	2·78	3·93	5·01	6·03	2·50	2·59
40	5	5·79	4·43	3·34	4·44	5·48	6·47	2·39	2·52
45	5	6·08	5·33	4·26	5·35	6·37	7·34	2·30	2·46
50	5	6·38	5·90	4·85	5·91	6·92	7·87	2·22	2·42
55	5	6·70	6·21	5·17	6·21	7·20	8·14	2·16	2·39
60	5	7·04	6·91	5·88	6·92	7·91	8·85	2·12	2·39
65	5	7·39	7·54	6·49	7·55	8·55	9·50	2·09	2·41
70	5	7·76	8·11	7·04	8·11	9·13	10·09	2·09	2·45
75	5	8·14	8·86	7·77	8·88	9·93	10·92	2·10	2·52
80	5	8·55	9·35	8·22	9·37	10·45	11·47	2·14	2·60
85	5	8·98	10·15	8·96	10·16	11·30	12·38	2·19	2·71
	Age at Entry, 30.								
35	5	5	2·18	1·33	2·34	2·02	2·98	2·79	2·81
40	5	5·25	3·61	2·82	3·76	3·46	4·35	2·65	2·70
45	5	5·51	4·66	3·89	4·80	4·51	5·37	2·53	2·62
50	5	5·79	5·32	4·58	5·47	5·18	6·02	2·42	2·55
55	5	6·08	5·68	4·96	5·82	5·54	6·35	2·33	2·50
60	5	6·38	6·48	5·77	6·62	6·35	7·16	2·26	2·47
65	5	6·70	7·18	6·46	7·31	7·04	7·85	2·23	2·47
70	5	7·04	7·78	7·06	7·92	7·65	8·46	2·21	2·49
75	5	7·39	8·60	7·85	8·74	8·45	9·29	2·21	2·54
80	5	7·76	9·11	8·35	9·25	8·96	9·82	2·24	2·62
85	5	8·14	9·93	9·14	10·09	9·78	10·68	2·29	2·72
	Age at Entry, 40.								
45	5	5	3·64	2·99	3·73	3·15	3·86	2·74	2·76
50	5	5·25	4·39	3·78	4·47	3·94	4·60	2·61	2·66
55	5	5·51	4·85	4·25	4·92	4·41	5·04	2·49	2·58
60	5	5·79	5·78	5·20	5·85	5·35	5·97	2·41	2·53
65	5	6·08	6·54	5·97	6·62	6·12	6·74	2·35	2·51
70	5	6·38	7·23	6·65	7·30	6·80	7·42	2·31	2·51
75	5	6·70	8·11	7·52	8·18	7·67	8·30	2·30	2·55
80	5	7·04	8·63	8·04	8·71	8·19	8·83	2·32	2·61
85	5	7·39	9·50	8·88	9·58	9·04	9·71	2·35	2·70
	Age at Entry, 50.								
55	5	5	5·47	4·86	5·44	4·37	4·90	2·70	2·72
60	5	5·25	4·78	4·22	4·75	3·76	4·25	2·58	2·64
65	5	5·51	5·73	5·18	5·70	4·73	5·21	2·50	2·60
70	5	5·79	6·55	6·00	6·52	5·56	6·04	2·45	2·58
75	5	6·08	7·56	7·01	7·53	6·56	7·04	2·43	2·59
80	5	6·38	8·17	7·61	8·14	7·16	7·65	2·43	2·64
85	5	6·70	9·10	8·53	9·07	8·06	8·56	2·47	2·72
	Age at Entry, 60.								
65	5	5	7·87	7·13	7·50	5·58	5·98	3·38	3·41
70	5	5·25	5·50	4·84	5·17	3·46	3·82	3·29	3·37
75	5	5·51	6·78	6·13	6·46	4·75	5·11	3·24	3·36
80	5	5·79	7·54	6·87	7·21	5·48	5·85	3·24	3·40
85	5	6·08	8·67	7·99	8·33	6·57	6·93	3·27	3·50

	Age 20	Age 30	Age 40	Age 50	Age 60
Hypothetical Annual Surplus Margins, Columns 9 and 10	·176	·214	·253	·298	·439

Constant of Expenditure . . 3/- per £100 assured.

Memoranda as to Table IX.

Office A may be taken as a type of office having a scale of premiums *rather* below the standard "Select" office premiums assumed in this paper between ages 20 and 30 at entry, and after age 30 at entry, higher. The rate of expenditure considerably above the average. The rate of interest realized rather below the average deduced in this paper for 1894. Column 5 shows that in such an office a 30*s.* uniform bonus favours the younger entrants, and the policies of short or comparatively short duration.

Office B may be taken as a type of office having a scale of premiums *much* below the standard "Select" office premiums assumed in this paper between ages 20 and 45 at entry, and considerably higher after age 45. The rate of expenditure is slightly above the average. The rate of interest realized as nearly as possible the average. Column 7 shows that in such an office a 30*s.* uniform bonus favours the younger entrants and the policies of short or comparatively short duration.

Office C may be taken as a type of office having a scale of premiums higher than the standard "Select" office premiums assumed in this paper up to age at entry 45, and thereafter much lower. The rate of expenditure slightly above the average, and the rate of interest realized about the average. Column 9 shows that in such an office a 30*s.* uniform bonus favours the older entrants, and the policies of short or comparatively short duration.

Office D may be taken as a type of office having a scale of premiums below the standard at the younger ages at entry, and above the standard at the higher ages at entry. The rate of expenditure is rather above the average, and the rate of interest materially below the average. Column 11 shows that policies taken out at the younger ages at entry, and policies of long duration on the books, are favoured by a compound bonus of 30*s.*: indeed, the compound bonus system would seem to be unsuitable to such an office.

Office E may be taken as a type of office having premiums equal to, or nearly equal to, the standard office premiums assumed in this paper. The rate of expenditure rather below the average, and the realized rate of interest rather above the average. Column 11 would indicate that the miscellaneous profit in an office under such conditions, is divided nearly in the form of an

TABLE IX.

Illustrating for each £100 assured the combined effect of various Scales of Premium, realized Rates of Interest, and Rates of Expenditure on the Reversionary Bonuses resulting from Annual Surplus Margins and Excess Interest beyond the Valuation Rate. Valuation H^M *3 per-cent.* H^M *Mortality Experience.—Also the difference between such resulting Reversionary Bonuses and a Bonus of £1. 10s. per-cent per annum.*

Age	Uniform Bonus of £1. 10s. per-cent per annum	Compound Bonus of £1. 10s. per-cent per annum	UNIFORM BONUS: OFFICE A. Assumed realized Rate of Interest £3. 17s. 6d. per-cent	Difference between Results in Column 2 and Column 4	OFFICE B. Assumed realized Rate of Interest £4 per-cent	Difference between Results in Column 2 and Column 6	OFFICE C. Assumed realized Rate of Interest £4 per-cent	Difference between Results in Column 2 and Column 8	COMPOUND BONUS: OFFICE D. Assumed realized Rate of Interest £3. 12s. 6d. per-cent	Difference between Results in Column 3 and Column 10	OFFICE E. Assumed realized Rate of Interest £4. 2s. 6d. per-cent	Difference between Results in Column 3 and Column 12	OFFICE F. Assumed realized Rate of Interest £4. 2s. 6d. per-cent	Difference between Results in Column 3 and Column 14
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
							Age at Entry 20							
25	7·5	7·50	−1·31	8·81	−1·78	9·28	2·34	5·16	−1·09	8·59	3·31	4·19	3·76	3·74
30	7·5	8·06	2·40	5·10	2·04	5·46	6·02	1·48	2·42	5·64	7·07	·99	7·51	·55
35	7·5	8·67	2·87	4·63	2·62	4·88	6·47	1·03	2·69	5·98	7·63	1·04	8·06	·61
40	7·5	9·32	3·33	4·17	3·19	4·31	6·93	·57	2·97	6·35	8·20	1·12	8·63	·69
45	7·5	10·02	3·78	3·72	3·75	3·75	7·41	·09	3·22	6·80	8·79	1·23	9·21	·81
50	7·5	10·77	4·26	3·24	4·33	3·17	7·93	− ·43	3·49	7·28	9·44	1·33	9·86	·91
55	7·5	11·57	4·70	2·80	4·89	2·61	8·45	− ·95	3·74	7·83	10·12	1·45	10·53	1·04
60	7·5	12·44	5·14	2·36	5·43	2·07	8·99	−1·49	3·99	8·45	10·82	1·62	11·24	1·20
65	7·5	13·38	5·61	1·89	6·00	1·50	9·59	−2·09	4·26	9·12	11·60	1·78	12·03	1·35
70	7·5	14·38	6·06	1·44	6·55	·95	10·20	−2·70	4·50	9·88	12·40	1·98	12·84	1·54
75	7·5	15·46	6·51	·99	7·11	·39	10·85	−3·35	4·76	10·70	13·29	2·17	13·74	1·72
80	7·5	16·62	6·97	·53	7·66	− ·16	11·54	−4·04	5·01	11·61	14·21	2·41	14·67	1·95
85	7·5	17·86	7·42	·08	8·24	− ·74	12·28	−4·78	5·25	12·61	15·22	2·64	15·73	2·13
							Age at Entry 30							
35	7·5	7·50	− ·47	7·97	− ·88	8·38	1·99	5·51	− ·34	7·84	2·07	5·43	3·40	4·10
40	7·5	8·06	2·76	4·74	2·46	5·04	5·22	2·28	3·32	4·74	5·43	2·63	6·71	1·35
45	7·5	8·67	3·29	4·21	3·11	4·39	5·79	1·71	3·59	5·08	6·12	2·55	7·37	1·30
50	7·5	9·32	3·81	3·69	3·76	3·74	6·36	1·14	3·86	5·46	6·82	2·50	8·05	1·27
55	7·5	10·02	4·31	3·19	4·37	3·13	6·92	·58	4·12	5·90	7·52	2·50	8·74	1·28
60	7·5	10·77	4·81	2·69	4·99	2·51	7·51	− ·01	4·40	6·37	8·26	2·51	9·47	1·30
65	7·5	11·57	5·31	2·19	5·60	1·90	8·12	− ·62	4·67	6·90	9·01	2·56	10·23	1·34
70	7·5	12·44	5·79	1·71	6·18	1·32	8·73	−1·23	4·94	7·50	9·77	2·67	11·02	1·42
75	7·5	13·38	6·27	1·23	6·77	·73	9·37	−1·87	5·21	8·17	10·60	2·78	11·88	1·50
80	7·5	14·38	6·25	·75	7·36	·14	10·04	−2·54	5·18	8·90	11·44	2·94	12·77	1·61
85	7·5	15·46	7·22	·28	7·90	− ·40	10·72	−3·22	5·75	9·71	12·31	3·15	13·71	1·75

TABLE IX—(*continued*).

Age	Uniform Bonus of £1. 10s. per-cent per annum	Compound Bonus of £1. 10s. per-cent per annum	UNIFORM BONUS: OFFICE A. Assumed realized Rate of Interest £3. 17s. 6d. per-cent	Difference between Results in Column 2 and Column 4	OFFICE B. Assumed realized Rate of Interest £4 per-cent	Difference between Results in Column 2 and Column 6	OFFICE C. Assumed realized Rate of Interest £4 per-cent	Difference between Results in Column 2 and Column 8	COMPOUND BONUS: OFFICE D. Assumed realized Rate of Interest £3. 12s. 6d. per-cent	Difference between Results in Column 3 and Column 10	OFFICE E. Assumed realized Rate of Interest £4. 2s. 6d. per-cent	Difference between Results in Column 3 and Column 12	OFFICE F. Assumed realized Rate of Interest £4. 2s. 6d. per-cent	Difference between Results in Column 3 and Column 14
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
	Age at Entry 40													
45	7·5	7·50	·18	7·52	·16	7·34	1·26	6·24	1·01	6·49	2·04	5·46	2·79	4·71
50	7·5	8·06	3·06	4·44	3·16	4·34	4·22	3·28	3·60	4·46	5·10	2·96	5·83	2·23
55	7·5	8·67	3·67	3·83	3·88	3·62	4·92	2·58	3·93	4·74	5·93	2·74	6·65	2·02
60	7·5	9·32	4·27	3·23	4·61	2·89	5·62	1·88	4·27	5·05	6·75	2·57	7·46	1·86
65	7·5	10·02	4·85	2·65	5·30	2·20	6·30	1·20	4·59	5·43	7·57	2·45	8·28	1·74
70	7·5	10·77	5·38	2·12	5·95	1·55	6·95	·55	4·89	5·88	8·36	2·41	9·09	1·68
75	7·5	11·57	5·91	1·59	6·60	·90	7·62	− ·12	5·19	6·38	9·19	2·38	9·93	1·64
80	7·5	12·44	6·43	1·17	7·23	·17	8·27	− ·77	5·50	6·94	10·06	2·38	10·79	1·65
85	7·5	13·38	6·92	·58	7·84	− ·31	8·92	−1·42	5·78	7·60	10·87	2·51	11·64	1·74
	Age at Entry 50													
55	7·5	7·50	·71	6·79	2·08	5·42	·14	7·36	1·02	6·48	1·88	5·62	2·08	5·42
60	7·5	8·06	3·34	4·16	4·82	2·68	2·93	4·57	3·35	4·71	4·77	3·29	4·96	3·10
65	7·5	8·67	4·06	3·44	5·66	1·84	3·80	3·70	3·81	4·86	5·76	2·91	5·94	2·73
70	7·5	9·32	4·73	2·77	6·44	1·06	4·60	2·90	4·21	5·11	6·69	2·63	6·89	2·43
75	7·5	10·02	5·37	2·13	7·21	·29	5·36	2·14	4·60	5·42	7·61	2·41	7·81	2·21
80	7·5	10·77	5·95	1·55	7·95	− ·45	6·06	1·44	4·94	5·83	8·50	2·27	8·69	2·08
85	7·5	11·57	6·50	1·00	8·66	−1·31	6·72	·78	5·29	6·28	9·35	2·22	9·55	2·02
	Age at Entry 60													
65	7·5	7·50	2·28	5·22	7·00	·50	−1·86	9·36	4·05	3·45	1·42	6·08	5·11	2·39
70	7·5	8·06	4·76	2·74	9·58	−2·08	·83	6·67	6·15	1·91	4·23	3·83	7·90	·16
75	7·5	8·67	5·60	1·90	10·57	−3·07	1·85	5·65	6·64	2·03	5·40	3·27	8·88	− ·21
80	7·5	9·32	6·35	1·15	11·52	−4·02	2·68	4·82	7·09	2·23	6·43	2·80	9·69	− ·37
58	7·5	10·02	7·01	·46	12·48	−4·98	3·39	4·11	7·52	2·50	7·38	2·64	11·27	−1·25

	ANNUAL SURPLUS MARGINS PER £100 ASSURED: Age 20	Age 30	Age 40	Age 50	Age 60	CONSTANTS OF EXPENDITURE: £ s. d.
Office A	·106	·141	·183	·222	·407	0 5 10 per £100 assured
Office B	·074	·108	·174	·361	1·000	0 4 3 " "
Office C	·336	·327	·275	·151	− ·142	0 4 1 " "
Office D	·126	·215	·273	·276	·668	0 4 5 " "
Office E	·393	·328	·339	·331	·266	0 3 7 " "
Office F	·422	·429	·408	·352	·739	0 3 0 " "

N.B.—The annual surplus margins are derived by deducting from the premium charged (1) 5 per-cent of the premium, (2) the constant of expenditure, (3) the H^{M} 3 per-cent valuation premium. A procuration fee of £1 per £100 assured is assumed to be paid out of the first premium, materially affecting the bonus resulting at the end of the first five years.

equal reversionary bonus, which would coincide approximately with the results of the method suggested in this paper, for distribution of miscellaneous profit, where the profit from a favourable mortality experience forms the greater portion of it (*see* Table VIII, Column 10). The bonuses appear to be distributed in as equitable a form as is practicable, having regard to the assumed conditions.

Office F may be taken as a type of office having a scale of premiums, equal to the standard "Select" office premiums assumed in this paper at the younger ages at entry (20 to 30), but at the older ages higher. The rate of expenditure is much below the average, and the rate of interest realized rather above the average. The same remarks apply to this office as to *Office E*, with respect to the distribution of the miscellaneous profit. The bonuses appear to be distributed in as equitable a form as is practicable, having regard to the assumed conditions.

Discussion.

The President (Mr. A. J. Finlaison, C.B.) said the subject of the paper was one that had occupied much attention in the past and necessarily would continue to do so owing to the changing circumstances of the times. Probably that part of the paper referring to the revision of the rates of premium would attract most attention.

Mr. A. B. Adlard said that after perusing Tables III and IV of the paper one could not escape from the conclusion that some at least of the offices adopting the method of distribution under observation must have done so with little or no reference to its suitability having regard to the scale of premiums upon which the policies were issued. Taking one illustration in Table III (Office No. 3) it would be seen that in comparison with the "select" standard there was a divergence, ranging from minus 4*s*. 6*d*. at age 20, to plus 17*s*. 2*d*. at age 60, showing how very inequitable as between lives of different ages at entry this system of distribution must be for the office in question. With this rigid system of distribution it was not possible to do justice between two classes of assurers paying premiums under different scales; and it was, he thought, a very powerful argument in favour of the "contribution" method that it was possible to do absolute justice between such classes by first distributing amongst those paying premiums under a higher scale that portion of the surplus arising from the additional contributions paid by them. As to the influences affecting bonus prospects generally which the author had enumerated, all might not agree as to the order of their importance, which not only varied in different offices but also varied in the same offices at different periods. The author had placed the rate of mortality experienced,

as compared with that "expected", rather low down in the list, but with some offices the light rate of mortality had been one of the main sources of profit, and in one instance enabled an old office, whose valuations had from its inception been made upon a Northampton basis, to pass to an H^M 3 per-cent basis in a surprisingly short time. Having regard to the experience of recent years and to other circumstances, however, he did not think that profit from mortality could be reckoned upon in the future to such an extent as in the past, and in face of the fact that the rate of interest had been, and was still, falling, also remembering the liberal conditions and privileges now granted by most offices, they would he thought agree with the author in his third conclusion that they were probably approaching an era of diminished bonuses. The author referred to reversionary transactions as tending to the maintenance of an average rate of interest of over 4 per-cent, but he (Mr. Adlard) had come to the conclusion that to purchase them on such terms as were now often accepted would tend rather to diminish the average rate of interest than to increase it. He thought the Carlisle Table which was frequently employed in valuing reversions was totally unsuitable for the purpose, and he even doubted whether the mortality shown by the Government Annuitants Tables would show so light a rate of mortality as that which prevailed among the life-tenants in cases where the reversioners had sold their reversionary interests. He thought an investigation of the mortality amongst life-tenants in connection with reversions which had been bought by various offices would be of great value, and might have the effect to some extent of checking the extravagant prices which had been given of late. A very interesting point brought out in the paper which he had never before noticed was shown in the table headed "Bonus-Loading for one per-cent per annum", where it would be seen that the bonus-loadings were practically the same whatever table of mortality was used as the basis. After carefully considering the subject he thought that the reason for this was correctly given by the author. He quite agreed with the statement in the concluding paragraph of the paper that the system of distribution by a percentage lacked the elasticity of the "contribution" method, and it appeared to him that the only justification, from a scientific point of view, of its adoption was the fact there alluded to, namely: that in offices experiencing average results compound reversionary bonuses closely approximated to those produced by the "contribution" method.

Mr. G. F. Hardy said the author was certainly to be congratulated upon having attacked a difficult subject and thrown a good deal of light upon it. It was surprising to see what an amount of space the question of surplus distribution occupied in the volumes of the *Journal*. The first volume contained an important paper by Mr. Jellicoe (*J.I.A.*, i, 22,* 159) on the subject, and from that time till now it had been continually under discussion. The reason was no doubt that the question was incapable of exact solution; it was impossible to analyze strictly the sources of surplus, and those sources themselves were changing from time to time. The only attempt at exact solution was that made by the American Actuaries in what they called the "Contribution Method." That

was a highly scientific plan, but on the whole was not practicably workable—at all events in this country—the main objection being that the whole of the mortality fluctuations at individual ages reappeared in the shape of fluctuations in the bonus. That of course where an enormous number of lives were at risk was of less importance than where they were more limited. The modified contribution plan appeared on the whole to avoid that difficulty, but it had another objection—namely, that it was impossible to treat with perfect equity the miscellaneous sources of surplus. For the last few years the profits from mortality in many offices had almost equalled the profit from interest, and therefore a method which merely distributed scientifically the interest profit, and more or less haphazard the profit from mortality, could not be considered as a final solution of the problem. As to the importance of the mortality profit, he had drawn up a small table showing the profit arising from 1 per-cent surplus interest and 15 per-cent mortality profit. He found at 30, after 10 years' duration of a £100 policy, the profit from mortality was about 50 per-cent greater than from interest; after 20 years it became a little less (4*s.* as against 5*s.*); and after 30 years it was 5*s.* as against 8*s.* Coming however to older ages at entry, such as 50, there was a larger profit from mortality throughout the whole duration of the policy than there was from the 1 per-cent surplus interest; after 10 years it was 7*s.* against 3*s.* 6*d.*, and after 20 years about 30 per-cent higher. This showed that a method merely distributing the mortality profit by some rule-of-thumb process could not be considered as scientific or conclusive. In the present state of our knowledge it was not known what the true rates of mortality were, and that was a great argument in favour of adopting a simple method like the compound reversionary bonus method, instead of the much more complex "contribution" plan. Another point was worth consideration. There was a demand on the part of the public for immediate bonus, and a desire on the part of Actuaries for high reserves, and of course the two things were incompatible. The consequence was that every office which was doing an expanding business was more or less in the condition of having to find money out of its reserves on the old policies to provide reserves for the new business. That difficulty arose whatever ordinary system of bonus distribution was employed, but it might be to a large extent obviated if offices would only be sufficiently courageous to give no profits at all in respect of the first year of assurance. No profits were made during the first year, and, by starting bonus from the second year they would avoid a great many of the embarrassments which under the present system had to be faced. He agreed with the author as to the importance of premiums being computed with a view to the method of distribution in use; or, at all events, of there being some sort of harmony between the rates of premium and the mode of distributing profit. All would agree that "select" tables were the only sound basis upon which to form premiums. Whether Dr. Sprague's, the most recent tables, were the best he did not know that they could decide, and he rather inclined to the opinion that his (Dr. Sprague's) first set of tables (*J.I.A.*, xxi, 247) were superior to those published later. It was usual to throw the whole expense on to the loading

specially placed for that purpose on the premium, but no doubt a considerable proportion of the expense of large offices, especially those of old standing, was due to the cost of managing the investments. It would therefore be quite legitimate and in every way preferable to allot a certain proportion of the loading to the interest element. The formulas given by the author were numerous, and some of them looked formidable. It did not appear however that any of them could be reduced to simpler forms if they aimed at mathematically exact expressions, but he believed they might get sufficient accuracy for all practical purposes by shorter formulas. For instance, taking the question of the compound reversionary bonus as an illustration, if the bonuses were considered as compounding yearly (which would not produce a very great error in the estimate), for the value of 1 per-cent bonus the difference between the 3 per-cent and the 4 per-cent single premiums could be used. The 3 per-cent premium represented the value of an assurance increasing annually as a geometrical progression in the proportion of 1·04 over 1·03, and therefore that was approximately the value (at 4 per-cent) of the assurance and the future bonuses. The 4 per-cent premium represented the value of the assurance at entry, and the difference between the two would give the value of the compound reversionary bonus. The author's conclusions were, he thought, very well justified by the facts brought together in the paper—at all events as regards the main contention that the existing premiums, having regard to the particular method of bonus distribution, were not on the whole in accord with what they should be according to theory. Offices in which discrepancies were so large, as in some cases they were, must be affected thereby in the course of competition. If an office could ensure having a fair proportion of entrants at all ages and in all classes, it could be content if the average premium on the whole were sufficient for the risk, but through competition business tended to gravitate to those portions of the tables and to those ages at which the profit to the office would be least, and it was therefore necessary that the profit should be distributed proportionally through the tables, so that this selection might be as far as possible nullified.

Mr. J. Chisholm said he thought the companies were now in an entirely different position compared with the time when heavy loading was first adopted. The ordinary trader knew what was the cost of any article he manufactured and added to that a charge for rent and interest on capital, besides putting on something for profit. The position of Life Offices was different to this in two ways. Firstly, they were unable until a comparatively recent date to tell at all exactly what was the cost price of the article they dealt in, and secondly, if a charge were made for profits, it was practically all returned to the assured. They were now however able to calculate to a nicety what the cost of a bonus was, and it had always seemed to him an absurdity to charge for a bonus simply to return it to the assured. It was a most uneconomical mode of carrying on business because if they charged a premium specially for bonus they had the trouble of collecting it, had to pay some renewal commission, and an investment had to be found for it. It would be much better if

they were to limit the amounts they had to invest—especially seeing the great difficulty there was of finding suitable investments now—to the amounts they were compelled to charge to produce the sums assured. A more serious objection to making an addition that was not really required was that it encouraged offices to go to expense in getting business. The increment provided a fund on which drafts, in the shape of large commissions for procuring business, might be made without the company incurring serious danger for a long time, and human nature being what it was, there was a great temptation to be sanguine and incur these expenses hoping that the mortality would turn out better than was expected, or that some miscellaneous profit would recoup the company for the extra expense in buying business. He believed it would conduce to their carrying on business on a more moral basis if they were to sweep away the bonus altogether, except to a small extent as a margin against accidental variations. With regard to consistency in calculating the premiums, the author had pointed out the great inequalities there were as between the different ages at entry. At the higher ages there was, he (Mr. Chisholm) thought, a reason for the inequality which was fairly justifiable. At the principal assuring ages, from 20 to 40 or 50, there were a large number of entrants, and they could therefore base their experience upon a pretty wide basis; but when they came to higher ages, say 60, 65 and 70, in any one office there were very few new insurers at those ages, and therefore it was not sufficient to rest satisfied with the same loading as at other ages on premiums derived from a "select" table. They should make, as an additional safeguard, a special addition to compensate for the small number who had contributed to the experience.

The President, in proposing a vote of thanks to Mr. Andras (which was unanimously accorded), said with regard to Mr. Adlard's remarks as to the future rate of mortality, he might mention that no indication of a high rate of mortality had been shown among Government annuitants since the year in which influenza had caused so many deaths among aged male lives. No general material decrease in the vitality of life-annuitants had yet been observed. The question of the mortality of another class of persons, namely life-tenants who were connected with the reversions purchased by insurance companies, was of great interest. He hoped that this question would shortly be before the Council, and it would be a most valuable addition to the investigation into the mortality of insured and annuitant lives if the mortality of life-tenants was also investigated.

Mr. Andras, in acknowledging the vote of thanks, said he was much indebted to those gentlemen who had discussed the paper for throwing a good deal of new light upon various points in connection with this complex and difficult subject, and especially to Mr. Chisholm for his practical criticism as to the paradoxical practice of putting on a bonus-loading only to return it again to the assured as bonus. As he had said in the paper, it was possible to run alongside each other in the same prospectus a low-profit table of premiums and a high-profit table of premiums, and this might in some sort meet Mr. Chisholm's objection. He hoped that the suggestions made in his

paper might lead to a more careful consideration of the equity of scales of premiums in various offices in relation to their systems of bonus distribution and the present circumstances of Life Assurance business.

ACTUARIAL NOTES.

I. *Graduation Formulas.* *By* G. F. Hardy F.I.A., *Actuary of the English and Scottish Law Life Assurance Association.*

Consequent upon the reading of Mr. J. A. Higham's original paper on the Graduation of Mortality Tables (*J.I.A.*, xxiii, 335) I worked out some general formulas similar in principle to those discussed by Mr. Levine in the January number of the *Journal* (*J.I.A.*, xxxii, 290), but expressed in somewhat different form, of which it may be of interest to give a short account. It is throughout assumed that the fourth and higher differences of the function to be graduated are zero.

If we express the operation of summing n successive terms of such a function, of which u_0 is the central term, by the symbol $[n]$, we shall have, using the ordinary central difference formulas,

$$[n]u_0 = u_{-\frac{n-1}{2}} + u_{-\frac{n-3}{2}} + \ldots + u_{\frac{n-3}{2}} + u_{\frac{n-1}{2}}$$

$$= nu_0 + \frac{n^3-n}{24}b_0 = n\left\{1 + \frac{24}{n^2-1}b\right\}u_0 \quad . \quad . \quad . \quad (1)$$

where $b_0 = \Delta^2 u_{-1}$ and the symbol b may be conveniently taken to represent the symbol of operation giving $b(u_0) = b_0$.

As fourth and higher differences of u_0 are zero, it follows that all terms involving b^2 and higher powers are to be neglected. Henee

$$\left\{[l]\,[m]\,[n] \ldots {\scriptstyle (t \text{ terms})}\right\}u_0 = lmn \ldots \left\{1 + \frac{l^2+m^2+n^2 \ldots -t}{24}b\right\}u_0 \;. \quad (2)$$

and

$$\left\{1 - \frac{l^2+m^2+n^2+ \ldots -t}{24}b\right\}\left\{[l]\,[m]\,[n] \ldots\right\}u_0 = (lmn \ldots)u_0 \quad . \quad (3)$$

where, of course, the order in which the operations on the left hand side are performed is immaterial.

Put into words this formula may be read thus:

If the successive values of a given function of the third or a lower order be summed in groups of $[l]$, these results again in

groups of $[m]$, &c., and from the final values so obtained be deducted $\dfrac{(l^2-1)+(m^2-1)+\&c.}{24}$ times the central second differences, the resulting quantities will be $l.m$ . . times the original values of the function.

To take an example, let $l=m=n=5$, then we have (writing

$$[5]^3=[5][5][5])$$

$$(1-3b)[5]^3u_0=125u_0 \text{ or } \frac{(1-3b)[5]^3u_0}{125}=u_0$$

which will be found to correspond to Mr. Woolhouse's formula; giving the following rule for obtaining his results: "Sum three "times successively in groups of five, subtract three times the "central second differences of the results so obtained, divide by "125, and the result is the adjusted value of u_0."

If we substitute for the expression $(1-3b)u_0=u_0-3b_0$ the equivalent expression (as far as 3rd differences) $u_0-(b_{-1}+b_0+b_1)$, in other words, if in lieu of subtracting from each term resulting from the successive summations three times the central second difference, we subtract the sum of three consecutive central second differences, we arrive at Mr. J. A. Higham's shorter formula.

Further, if we take the operations represented by the symbols:

$$x[m]-[n]=(xm-n)u_0+\frac{x(m^3-m)-(n^3-n)}{24}b_0 \quad . \quad . \quad (4)$$

$$=(xm-n)\left\{1+\frac{x(m^3-m)-(n^3-n)}{24(xm-n)}b\right\}u_0 \quad . \quad (5)$$

(where x is merely a numerical coefficient), then, the coefficient of b may be made to assume any value we please, + or −. Hence by properly selecting the values of x, h, k, &c., an indefinite number of expressions, correct to 3rd differences, may be found of the form

$$[l][m][n] \ . \ . \ . \left\{x[h]-[k]\right\}u_0=lmn \ . \ . \ . \ (xh-k)u_0 \quad . \quad (6)$$

The different values, however, of h, k, l, m, &c., that can be advantageously used in practice are somewhat restricted in range, and it is more convenient to consider a few numerical examples than to elaborate the algebra of the subject. The

following table gives for values of n from 1 to 20 the values of n^2-1 and n^3-n, which enable us to employ in practice any of the above formulas.

Table of Coefficients for Formula $[n]=n\left(1+\frac{n^2-1}{24}\right)u_0$.

n	* n^2-1	* n^3-n	n	* n^2-1	* n^3-n
1	0	0	11	120	1,320
2	3	6	12	143	1,716
3	8	24	13	168	2,184
4	15	60	14	195	2,730
5	24	120	15	224	3,360
6	35	210	16	255	4,080
7	48	336	17	288	4,896
8	63	504	18	323	5,814
9	80	720	19	360	6,840
10	99	990	20	399	7,980

* N.B.—The coefficients n^2-1, n^3-n, have in all cases the denominator 24, but it is more convenient to omit this in the table.

For example we have

$$10[1]-3[3]=10u_0-3\left(3u_0+\frac{24}{24}b_0\right)=u_0-72.\frac{b_0}{24}=u_0-3b_0.$$

Now we see from the table that the operation

$$[5]=5\left(1+24\cdot\frac{b}{24}\right)u_0=5(1+b)u_0,$$

hence $\qquad [5]^3=125(1+3b)u_0.$

Thus the operations $[5]^3$ will eliminate the terms involving b introduced in the operations $10[1]-3[3]$, and we shall have

$$[5]^3(10[1]-3[3])=125(1+3b)(1-3b)u_0=125u_0.$$

Also, since

$$10[1]-3[3]=10u_0-3(u_{-1}+u_0+u_1)=-3u_{-1}+7u_0-3u_1$$

$$[5]^3(-3u_{-1}+7u_0-3u_1)=125u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is another form of Woolhouse's formula.

Again, $\qquad 2[3]-[5]=-u_{-2}+(u_{-1}+u_0+u_1)-u_{+2}$

$$=2\left(3u_0+24\frac{b_0}{24}\right)-\left(5u_0+120\frac{b_0}{24}\right)=u_0-3b_0 \quad . \quad . \quad (8)$$

in which the term involving b_0 can also be eliminated by the operations $[5]^3$, *i.e.*, three successive summations in 5's, giving another form of Higham's formula, namely,

$$\frac{1}{125}[5]^3\{2[3]-[5]\}=u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

or

$$\frac{1}{125}[5]^3\{u_0+(\Delta u_{-2}-\Delta u_1)\}=u_0 \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

which latter form gives a convenient working formula.

It will be seen from an inspection of the values of n^2-1 in the above table that in lieu of the operations $\frac{[5]^3}{125}$ we might employ the double operation $\frac{[7]\,[5]}{35}$, or the four operations $\frac{[7]\,[3]^3}{189}$, or the six operations $\frac{[5]^2\,[4]\,[2]^3}{800}$. For example, we should have

$$\frac{[7]\,[5]}{35}\{u_0+(\Delta u_{-2}-\Delta u_1)\}=u_0$$

or writing the terms in order,

$$\frac{-u_{-7}-u_{-6}+2u_{-4}+3u_{-3}+5u_{-2}+6u_{-1}+7u_0+6u_1\,\&c.}{35}=u_0 \quad . \qquad (11)$$

a simple formula giving results somewhat smoother than that of Woolhouse, but less smooth than Higham's.

Again, the three operations $\frac{[4]\,[5]\,[6]}{120}$ may be taken as approximately equal to $\frac{[5]^3}{125}$.

As will be seen from the values in the second column of the table,

$$\frac{[4]\,[5]\,[6]}{120}u_0=\left(1+\frac{15+24+35}{24}b\right)u_0=\left(1+\frac{74}{24}b\right)u_0,$$

hence

$$\frac{[4]\,[5]\,[6]}{120}\left(1-\frac{72}{24}b\right)u_0=\left(1+\frac{1}{12}b\right)u_0 \quad . \quad . \qquad (12)$$

a result differing from u_0 by $\frac{1}{12}$ of its central second difference, an error of no great importance in practice. Hence, we may write approximately (see formula 10):

$$\frac{[4]\,[5]\,[6]}{120}\{u_0+(\Delta u_{-2}-\Delta u_1)\}=u_0 \quad . \quad . \quad . \qquad (13)$$

agreeing with the formula of graduation given in my paper on Friendly Societies (*J.I.A.*, xxvii, 277).

As a final illustration take the following expression:

$$\{[3]+[5]-[7]\}u_0=(u_{-1}+u_0+u_1)-(u_{-3}+u_3).$$

We have from the above table:

$$[3]=\quad 3u_0+\ \ 24\cdot\frac{b_0}{24}$$

$$[5]=\quad 5u_0+120\cdot\frac{b_0}{24}$$

$$-[7]=-7u_0-336\cdot\frac{b_0}{24}$$

$$\therefore\quad [3]+[5]-[7]=u_0-192\frac{b_0}{24}.$$

Looking in the second column of the table for values of n^2-1 that together will make up the numerator 192, we find this can be done (among various alternatives) by combining the summations corresponding to $m=5$ and $n=13$. Hence we have (dividing by the product mn).

$$\frac{[5][13]}{65}\{[3]+[5]-[7]\}u_0=\frac{[5][13]}{65}\{(u_{-1}+u_0+u_1)-(u_{-3}+u_3)\}$$

$$=\left(1+192\frac{b}{24}\right)\left(1-192\frac{b}{24}\right)u_0$$

$$=u_0,$$

giving a formula embracing 23 ungraduated values of u_0 and suitable for the graduation of a curve, not changing greatly in character over this range, where the original values are somewhat irregular. Or the operations $[12][6][3][2]^2$ might be employed.

To refer to even a small proportion of the various formulas more or less suitable for the graduation of mortality tables that may be written down by the aid of the table, would expand this short note into a paper of some dimensions. It will be sufficient to point out certain general principles, which are, in fact, pretty obvious.

The irregularities in a mortality curve consist not only of accidental deviations from the mean curve at individual ages—small ripples as it were—but also of longer waves, due to the grouping of the smaller irregularities. Formulas of graduation

involving in the summations only a small number of consecutive values of the function, will deal effectively only with the former class of irregularities; formulas embracing a wider range of values must be employed if it is desired to smooth down the larger irregularities.

It is also to be noticed that a formula based on a series of summations, such as are represented by the symbols [4], [5], [6], will generally give a smoother curve than one based on the approximately equivalent operations $[5]^3$.

It is convenient to have some test (other than that of actual trial) of the smoothness of the curve that will be brought out by any given formula, though any test must of course be more or less arbitrary, and can only be relative.

It has been assumed throughout that 4th differences are zero, and in general, if all accidental irregularities were effectually removed, the third (or even the second) differences of such functions as q_x or d_x should be small. If we assume that each of the ungraduated values upon which our graduated results are based are affected by a similar probable error, we shall then be able to express the probable error in the graduated values, and in the successive orders of differences of both the ungraduated and graduated curves. This will be best seen by an illustration, and for this purpose we may select Mr. Woolhouse's well known formula. Here, writing u'_0 for the adjusted value of u_0,

$$u'_0 = \tfrac{1}{125}\{-3(u_{-7}+u_7)-2(u_{-6}+u_6)+3(u_{-4}+u_4)+7(u_{-3}+u_3)$$
$$+21(u_{-2}+u_2)+24(u_{-1}+u_1)+25u_0\} \quad . \quad (14)$$

and if each of the quantities u_{-7}, u_{-6}, &c., on the right-hand side are affected by a probable error of e, the probable error in u'_0 will be $\frac{1}{125}$ of this quantity multiplied by the square root of the sum of the squares of the coefficients -3, -3, -2, &c.:

$$=\frac{\sqrt{2(3^2+2^2+3^2+7^2+21^2+24^2)+25^2}}{125}\cdot e$$

$$=\frac{\sqrt{2{,}801}}{125}e=\frac{53}{125}e \text{ approximately.*}$$

Hence the accidental errors in the original values of u_x would be reduced by graduation in this proportion; or to about the

* [This result was also deduced by the late W. S. B. Woolhouse (*J.I.A.*, xxvi, 422).—ED. *J.I.A.*]

value they would have in an ungraduated experience of *five* times the magnitude (since $\frac{125^2}{2801}=5$ nearly). The *smoothness* of the curve, as tested say by the irregularities in the 3rd differences, would however be much greater than that of an ungraduated curve based upon the larger experience (this because the errors in the ungraduated values of u_x are independent, but in the graduated values they are not so, most of the quantities upon which the graduated value of u_x depends, reappearing with slightly altered coefficients in the graduated value of u_{x+1}, &c.).

If we write down in order the coefficients simply, in Woolhouse's formula, together with their differences, we get (writing as before u'_0 for the adjusted value of u_0)

Coeffts. in $u'_0=\frac{1}{125}[-3, -2, 0, 3, 7, 21, 24, 25, 24, 21, 7, 3, 0, -2, -3]$

" $\Delta u'_0=\frac{1}{125}[-3, 1, 2, 3, 4, 14, 3, 1, -1, -3, -14, \text{\&c.}]$

" $\Delta^2 u'_0=\frac{1}{125}[-3, 4, 1, 1, 1, 10, -11, -2, -2, -2, -11, 10, \text{\&c.}]$

" $\Delta^3 u'_0=\frac{1}{125}[-3, 7, -3, \,..\, 9, -21, 9, \,..\, -3, 7, -3, \,..\, 9, -21, 9]$

Comparing the coefficients thus obtained for $\Delta^3 u'_0$ with those for the ungraduated value, namely, $1, -3, 3, -1$, the relative errors in the graduated and ungraduated values will be seen to be in the ratio of

$$\tfrac{1}{125}\,.\,\sqrt{3^2+7^2+3^2} : 1,$$

$$\text{say}\quad \tfrac{1}{15} : 1.$$

If we apply a similar test to the formula (11), we get

Coeffts. in $u'_0=\frac{1}{35}[-1, -1, 0, 2, 3, 5, 6, 7, 6, 5, 3, 2, 0, -1, -1]$

" $\Delta u'_0=\frac{1}{35}[-1, 0, 1, 2, 1, 2, 1, 1, -1, -1, \text{\&c.}]$

" $\Delta^2 u'_0=\frac{1}{35}[-1, 1, 1, 1, -1, 1, -1, 0, -1, 0, -1, 1, -1, 1, 1, 1, -1]$

" $\Delta^3 u'_0=\frac{1}{35}[-1, 2, 0, 0, -2, 2, -2, 1, -1, 1, -1, 2, -2, 2, 0, 0, -2, 1]$

and the relative errors in the graduated and ungraduated values of $\Delta^3 u_0$ will be as

$$\tfrac{1}{35}\sqrt{6+8\times 2^2} : \sqrt{1+3^2+3^2+1}$$
$$=\tfrac{1}{35}\sqrt{\tfrac{38}{20}} : 1$$
$$\text{say } \tfrac{1}{25} : 1 .$$

With Mr. Higham's shorter formula, this ratio would become about $\frac{1}{56} : 1$; or, in formula (13), about $\frac{1}{100} : 1$.

II. *On Formulas for Graduation by Summation. By* RALPH TODHUNTER, M.A., F.I.A., *of the National Life Assurance Society.*

The following Note is submitted by the writer in the belief that a short review of the theory of graduation by summation will be of some general interest on account of the historical importance of that method of graduation; the application of the method to the graduation of mortality tables has been shown by Dr. Sprague to be open to some objections, and may be considered as limited to cases in which Makeham's (or some similar) formula, or a graphic method, cannot be satisfactorily applied.

I. IN Mr. J. A. Higham's two papers in the 25th volume of the *Journal* (*J.I.A.*, xxv, 15, 245), it is shown how the result of summing an odd number of terms p at a time, the terms of the resulting series q at a time, and so on until the terms are all collected into a single term ($S_{pqr\ldots}$), may be expressed in terms of the first term of the original series and its differences, and thence how the central term of the original series may be expressed correctly to the third order of differences in terms of $S_{pqr\ldots}$ and $S_{p'q'r'\ldots}$ where $\Sigma p - n = \Sigma p' - n' = N - 1$, n and n' being the numbers of operations in $S_{pqr\ldots}$ and $S_{p'q'r'\ldots}$ respectively, and N being the number of terms (the same, it will be observed, in each case) of the original series included in each summation. In the last number of the *Journal* (*J.I.A.*, xxxii, 290), Mr. Levine has shown that the process of determining the coefficients of the differences in the expression for S may be simplified by employing the methods of symmetrical algebra.

The relation between S and the central term of the original series may be directly and more readily obtained by the following method:

(1) Suppose p, q, r, &c., are all *odd*, then the number of terms in each of the series obtained by the successive summations

will be *odd* (the effect of summing p at a time being to reduce the number of terms by $p-1$, an *even* number since p by hypothesis is *odd*). Each such series will have a central term, and the successive central terms will lie in a horizontal row beginning with u_0 and ending with S. The central term of the series obtained by summation in p's will be=

$$\left[(1+\Delta)^{-\frac{p-1}{2}}+\ldots+(1+\Delta)^{\frac{p-1}{2}}\right]u_0;$$ the central term of the

next series will be=

$$\left[(1+\Delta)^{-\frac{q-1}{2}}+\ldots+(1+\Delta)^{\frac{q-1}{2}}\right]\left[(1+\Delta)+\ldots+(1+\Delta)^{\frac{p-1}{2}}\right]u_0,$$

and so on, leading finally to,

$$S_{pqr\ldots}=\Pi^*_{pqr\ldots}\left[(1+\Delta)^{-\frac{p-1}{2}}+\ldots+(1+\Delta)^{\frac{p-1}{2}}\right]u_0$$

Expanding, and taking each pair of terms $(1+\Delta)^{-m}$ and $(1+\Delta)^m$ together:

$$S_{pqr\ldots}=\Pi_{pqr\ldots}\left[1+\Sigma_1^{\frac{p-1}{2}}\left(2+m^2\overline{\Delta^2-\Delta^3}+\frac{m^4+11m^2}{12}\Delta^4-\frac{m^4+5m^2}{6}\Delta^5+\ldots\right)\right]u_0.$$

By inserting the values of the sums of the powers of the natural numbers, namely,

$$\Sigma_1^{\frac{p-1}{2}}m^2=\frac{p(p^2-1)}{24};\quad \Sigma_1^{\frac{p-1}{2}}m^4=\frac{p(p^2-1)(3p^2-7)}{480},$$

the above equation becomes

$$S_{pqr\ldots}=\Pi_{pqr\ldots}\left[p+\frac{p(p^2-1)}{24}\overline{\Delta^2-\Delta^3}+\frac{p(p^2-1)(p^2+71)}{1920}\Delta^4-\frac{p(p^2-1)(p^2+31)}{960}\Delta^5+\ldots\right]u_0$$

$$=pqr\ldots\left[1+\Sigma.\frac{p^2-1}{24}(\Delta^2-\Delta^3)+\Sigma\left(\frac{\overline{p^2-1}.\overline{p^2+71}}{1920}+\frac{\overline{p^2-1}.\overline{q^2-1}}{576}\right)\Delta^4-\Sigma\left(\frac{\overline{p^2-1}.\overline{p^2+31}}{960}+\frac{\overline{p^2-1}.\overline{q^2-1}}{288}\right)\Delta^5+\ldots\right]u_0;$$

* This symbol is used in symmetrical algebra (on the analogy of Σ) to denote the *product* of a number of symmetrical factors.

or
$$u_0=\frac{S_{pqr\ldots}}{pqr\ldots}-\frac{\Sigma p^2-n}{24}(\Delta^2-\Delta^3)u_0$$
$$-\frac{3\Sigma p^4+10(22-n)\Sigma p^2+10\Sigma p^2q^2+5n^2-218n}{5760}\Delta^4u_0$$
$$+\frac{3\Sigma p^4+10(10-n)\Sigma p^2+10\Sigma p^2q^2+5n^2-98n}{2880}\Delta^5u_0$$
$$+\text{terms involving differences of higher orders}\ldots(a)$$

n being the number of operations included in S. It will be observed that by the above method the coefficients of the *fourth* and *fifth* differences in the general expression for u_0 have been obtained, and that the coefficients of differences of higher orders may be obtained with comparatively little difficulty.

(2) Suppose, p, q, r, &c. are not all odd. The method of (1) cannot then be exactly followed, because the series obtained by the first summation in even numbers will include an *even* number of terms and will therefore not have a central term. Since, however, the number of terms in the original series is odd, and the last summation gives a single term, namely S, it is obvious that summations in even numbers must occur in pairs, and the order of operation being immaterial, any pair of even summations, say the q and r summations may be taken together. The summation in q's will give a term lying just above the central row and $=[(1+\Delta)^{-\frac{q-1}{2}}+\ldots+(1+\Delta)^{\frac{q-1}{2}}](1+\Delta)^{-\frac{1}{2}}v_0$ where v_0 is the central term of the preceding series. The summation in r's will restore a central term which will be $=[(1+\Delta)^{-\frac{r-1}{2}}+\ldots$

$$+(1+\Delta)^{\frac{r-1}{2}}](1+\Delta)^{\frac{1}{2}}[(1+\Delta)^{-\frac{q-1}{2}}+\ldots+(1+\Delta)^{\frac{q-1}{2}}](1+\Delta)^{-\frac{1}{2}}v_0$$
$$=\{(1+\Delta)^{-(r-1)}+\ldots+1\}\{(1+\Delta)^{-(q-1)}+\ldots1\}(1+\Delta)^{\frac{q+r}{2}-1}v_0.$$

Now the indices of $(1+\Delta)$ in this expression are all integral, and are the same as if q and r had both been odd. It appears, therefore, that the result obtained by a pair of summations in even numbers is of precisely the same *form* as that obtained by a pair of summations in odd numbers. Hence, the result obtained in (1) on the assumption that p, q, r, &c., are all odd, holds good for any integral values of p, q, r, &c., consistent with the process under consideration.

II. The general expression for u_0 in terms of two summations and of its own differences of higher orders than the third is found in the following manner:

$$\text{From } (a)\ u_0 = \frac{S_{pqr\ldots}}{pqr\ldots} - \frac{\Sigma p^2 - n}{24}(\Delta^2 - \Delta^3)u_0 - \frac{k + 220(\Sigma p^2 - n)}{5760}\Delta^4 + \frac{k + 100(\Sigma p^2 - n)}{2880}\Delta^5 + \ldots$$

where k is written for $3\Sigma p^4 + 10\Sigma p^2q^2 + 5n^2 + 2n(1 - 5\Sigma p^2)$, whence

$$\frac{u_0}{\Sigma p^2 - n} = \frac{S_{pqr\ldots}}{(\Sigma p^2 - n)pqr\ldots} - \frac{\Delta^2 - \Delta^3}{24}u_0 - \left(\frac{k}{5760(\Sigma p^2 - n)} + \frac{11}{288}\right)\Delta^4 u_0 + \left(\frac{k}{2880(\Sigma p^2 - n)} + \frac{10}{288}\right)\Delta^5 u_0 + \ldots$$

Similarly,

$$\frac{u_0}{\Sigma p'^2 - n'} = \frac{S_{p'q'r'\ldots}}{(\Sigma p'^2 - n')p'q'r'\ldots} - \frac{\Delta^2 - \Delta^3}{24}u_0 - \left(\frac{k'}{5760(\Sigma p'^2 - n')} + \frac{11}{288}\right)\Delta^4 u_0 + \left(\frac{k'}{2880(\Sigma p'^2 - n')} + \frac{10}{288}\right)\Delta^5 u_0 + \ldots$$

Subtracting,

$$\left(\frac{1}{\Sigma p^2 - n} - \frac{1}{\Sigma p'^2 - n'}\right)u_0 = \frac{S_{pqr\ldots}}{(\Sigma p^2 - n)pqr\ldots} - \frac{S_{p'q'r'\ldots}}{(\Sigma p'^2 - n')p'q'r'\ldots} - \frac{1}{5760}\left(\frac{k}{\Sigma p^2 - n} - \frac{k'}{\Sigma p'^2 - n'}\right)(\Delta^4 - 2\Delta^5)u_0 + \ldots \quad (\beta)$$

The form in which this result has been obtained shows that the two summations need not include the same number of terms of the original series. This fact is incidentally involved in Mr. J. A. Higham's deduction of a summation formula for Mr. Woolhouse's method of graduation (*J.I.A.*, xxxi, 332, where two summations involving 13 and 15 terms respectively are employed), but it does not appear to have been contemplated in his original demonstration of the theory, and in the various formulas he has proposed the summations in each case extend over the same number of terms. The following table of these formulas, showing the number of terms involved in each and the error to fifth differences (calculated from β) may be of interest:

Number of Terms of Original Series involved	Formula for u_0.	Error to Fifth Differences
7	$\frac{1}{5}S_{5.3}-\frac{2}{7}S_7$	$-\frac{2}{5}(\Delta^4-2\Delta^5)$
9	$\frac{1}{10}S_{5.5}-\frac{1}{6}S_9$	$-(\Delta^4-2\Delta^5)$
11	$\frac{6}{100}S_{5.5.3}-\frac{1}{10}S_{5.7}$	$-\frac{19}{10}(\Delta^4-2\Delta^5)$
15	$\frac{16}{1000}S_{5.5.5.3}-\frac{1}{35}S_{5.5.7}$	$-\frac{22}{5}(\Delta^4-2\Delta^5)$
17	$\frac{64}{10000}S_{5.5.5.5}-\frac{1}{75}S_{5.5.9}$	$-6(\Delta^4-2\Delta^5)$
"	$\frac{16}{1000}S_{5.5.5.4.2}-\frac{24}{1000}S_{5.5.5.5}$	$-\frac{32}{5}(\Delta^4-2\Delta^5)$

By combination of any two of the above formulas a formula correct to fifth differences may be obtained. For example

$$6u_0=\frac{6}{10}S_{5.5}-S_9+6(\Delta^4-2\Delta^5)u_0+\ldots$$

$$u_0=\frac{64}{10000}S_{5.5.5.5}-\frac{1}{75}S_{5.5.9}+6(\Delta^4-2\Delta^5)u_0+\ldots$$

$$\therefore\quad u_0=\frac{3}{25}S_{5.5}-\frac{64}{50000}S_{5.5.5.5}-\frac{1}{5}S_9+\frac{1}{375}S_{5.5.9}+\ldots$$

It will be observed that four summations enter into this expression, but u_0 may in general be expressed correctly to the fifth order of differences in terms of three summations. From (a)

$$u_0-\frac{S_{pqr\ldots}}{pqr\ldots}+\frac{\Sigma p^2-n}{24}\left(\Delta^2-\Delta^3+\frac{11}{12}\Delta^4-\frac{5}{6}\Delta^5\right)u_0$$
$$+\frac{k}{5760}(\Delta^4-2\Delta^5)u_0+\ldots=0$$

Similarly $u_0-\dfrac{S_{p'q'r'\ldots}}{p'q'r'\ldots}+\dfrac{\Sigma p'^2-n'}{24}\left(\Delta^2-\Delta^3+\dfrac{11}{12}\Delta^4-\dfrac{5}{6}\Delta^5\right)u_0$

$$+\frac{k'}{5760}(\Delta^4-2\Delta^5)u_0+\ldots=0$$

and $$u_0 - \frac{S_{p''q''r''\dots}}{p''q''r''\dots} + \frac{\Sigma p''^2 - n''}{24}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5\right)u_0$$
$$+ \frac{k''}{5760}(\Delta^4 - 2\Delta^5)u_0 + \dots = 0$$

The elimination of the differences from these three equations gives

$$\begin{vmatrix} u_0 - \dfrac{S_{pqr\dots}}{pqr\dots} & \Sigma p^2 - n & k \\ u_0 - \dfrac{S_{p'q'r'\dots}}{p'q'r'\dots} & \Sigma p'^2 - n' & k' \\ u_0 - \dfrac{S_{p''q''r''\dots}}{p''q''r''\dots} & \Sigma p''^2 - n'' & k'' \end{vmatrix} = 0 \dots (\gamma)$$

as the equation expressing u_0 correctly to fifth differences in terms of three summations.

For integral values of $p, q, r, \dots p', q', r' \dots n$ and n', satisfying the equation

$$\frac{k}{\Sigma p^2 - n} = \frac{k'}{\Sigma p'^2 - n'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

it will be seen by reference to (β) that u_0 may be expressed correctly to the fifth order of differences in terms of *two* summations.

Equation (δ) does not appear to admit of general integral solution, but as an example of its application the special case may be taken in which

$$q = r = \dots = p\,; \quad q' = r' = \dots = p'.$$

It then reduces to

$$5np^2 - 2p^2 - 5n = 5n'p'^2 - 2p'^2 - 5n',$$

or, putting $p^2 - 1 = \pi$,

$$(5n - 2)\pi = (5n' - 2)\pi' \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

of which integral solutions may be found by inspection. Taking as an example $\pi = 48$; $\pi' = 8$ (whence $p = 7$; $p' = 3$) (ϵ) becomes

$$n' = 6n - 2,$$

which is satisfied by $n = 1$; $n' = 4$.

These values give $\frac{1}{27}$ $S_{3.3.3.3} - \frac{2}{7}$ S_7 as a summation formula for u_0 correct to *fifth* differences. Expressed in the terms of the original series, this formula is

$$\cdot\dot{4}1798\dot{9}u_0 + \cdot\dot{3}0687\dot{8}(u_{-1}+u_1) + \cdot\dot{0}8465\dot{6}(u_{-2}+u_2)$$
$$-\cdot\dot{1}3756\dot{6}(u_{-3}+u_3) + \cdot\dot{0}3\dot{7}(u_{-4}+u_4).$$

It is unnecessary to give further examples of the construction of formulas correct to third or fifth differences, and involving two or more summations of the same or different numbers of terms of the original series, but the formulas corresponding to Mr. Woolhouse's method of graduation may be mentioned on account of the historical interest of that method. The practicability of applying Mr. Woolhouse's method by a process of columnar summation appears to have been first pointed out by Mr. G. F. Hardy in the discussion that followed the reading of Mr. J. A. Higham's paper on "The Adjustment of Mortality Tables" (*J.I.A.*, xxiii, 351); in a contribution appearing in the number of the *Journal* in which this discussion is reported, Mr. T. G. Ackland (*J.I.A.*, xxiii, 352) showed that Mr. Woolhouse's coefficients were reproduced by the ordinary summation formula $\frac{1}{125}(S_{5.5.5}+9S_5-3S_{15})$, and subsequently Mr. J. A. Higham (*J.I.A.*, xxxi, 319), showed that the same result might be obtained by means of the formula $\frac{1}{125}[10\,S_{5.5.5} - 3S_{5.5.5.3}]$. The first formula may be verified as correct to third differences by evaluation of the summations by (a), and the second by direct application of (β). It will be found that $\frac{1}{125}(S_{5.5.5}+9S_5-3S_{15})$

$$=\frac{1}{125}(10S_{5.5.5}-3S_{5.5.5.3}) = u_0 - \frac{27}{5}(\Delta^4 - 2\Delta^5) + \ldots.$$

For purposes of record the various formulas involving second-difference errors that have been used, or suggested for use, in graduation of mortality tables, are given in the following table with the corresponding errors to fifth differences. Nos. 1 and 2 are Mr. John Finlaison's (*J.I.A.*, xxi, 50); No. 3 is attributed to Mr. A. G. Finlaison; No. 4 will be easily

recognized as the formula given by Mr. Charles Ansell, Jun., in his *Statistics of Families* in the form $\cdot 17\dot{3}u_0 + \cdot 16(u_{-1}+u_1) + \cdot 12(u_{-2}+u_2) + \cdot 08(u_{-3}+u_3) + \cdot 04(u_{-4}+u_4) + \cdot 01\dot{3}(u_{-5}+u_5)$; No. 5 was given by Mr. G. F. Hardy in the Messenger Prize Essay on Friendly Societies (*J.I.A.*, xxvii, 245), as an improvement in smoothness and simplicity on then existing formulas, and as involving an error of only one-twelfth of the central second difference:

Number	Number of Terms of Original Series involved	Formula	Error to Fifth Difference
1	7	$\frac{1}{15}S_{3.5}$	$\frac{4}{3}(\Delta^2-\Delta^3)+\frac{28}{15}\Delta^4-\frac{12}{5}\Delta^5$
2	9	$\frac{1}{25}S_{5.5}$	$2(\Delta^2-\Delta^3)+\frac{17}{5}\Delta^4-\frac{24}{5}\Delta^5$
3	13	$\frac{1}{125}S_{5.5.5}$	$3(\Delta^2-\Delta^3)+\frac{33}{5}\Delta^4-\frac{51}{5}\Delta^5$
4	11	$\frac{1}{75}S_{5.5.3}$	$\frac{7}{3}(\Delta^2-\Delta^3)+\frac{22}{5}\Delta^4-\frac{97}{15}\Delta^5$
5	17	$\frac{1}{60}S_{3.4.5.6}-\frac{1}{120}S_{4.5.5.6}$	$\frac{1}{12}(\Delta^2-\Delta^3)-\frac{257}{40}\Delta^4+\frac{194}{15}\Delta^5$

III. The expression for S in terms of u_0 and its differences assumes an interesting and somewhat more symmetrical form in terms of the central differences of u_0. If in the expression $\Pi_{pqr\ldots}\left[(1+\Delta)^{\frac{-p-1}{2}}+\ldots+(1+\Delta)^{\frac{p-1}{2}}\right]u_0$ the operating factors are taken in pairs, as in (I), but converted in working into central differences it will be found that

$$S_{pqr\ldots}=\Pi\left[1+\Sigma_1^{\frac{p-1}{2}}\left(2+m^2b_0+\frac{m^2(m^2-1)}{12}d_0 + \frac{m^2(m^2-1)(m^2-4)}{360}f_0+\ldots\right)\right]u_0$$

the coefficients of the intermediate central differences disappearing. Proceeding as before, this equation reduces to

$$S_{pqr\ldots}=pqr\ldots\left[1+\frac{\Sigma p^2-n}{24}b_0+\frac{k-20(\Sigma p^2-n)}{5760}d_0+\ldots\right]u_0.$$

where k has the same meaning as in (I).

This result may also be deduced directly from (a).

In this form the equation exhibits very clearly the nature of the error introduced by summing and averaging. Thus, to the fifth central difference (the coefficient of e_0 in the above equation being zero), any summation formula $S_{pqr\ldots}\,pqr\ldots$ may be corrected by deducting certain multiples of the second and fourth central differences. Neglecting fourth differences and putting $q=r=\ldots=p$.

$$u_0=\frac{S_{ppp\ldots}}{p^n}-\frac{n(p^2-1)}{24}b_0$$

or, since $b_0=S_3-3u_0$, and the effect of operating on b_0 with S is to produce an error of magnitude d_0 only,

$$u_0=\frac{S_{ppp\ldots}}{p^n}\left[1-\frac{n(p^2-1)}{24}b_0\right]=\frac{S_{ppp\ldots}}{p^n}\left[1+\frac{n(p^2-1)}{8}-\frac{n(p^2-1)}{24}S_3\right]$$

Putting $p=5$ and $n=3$ this becomes

$$u_0=\frac{S_{5,5,5}}{125}[1-3b_0]=\frac{S_{5,5,5}}{125}[10-3S_3]$$

or $$\frac{1}{125}[10\,S_{5,5,5}-3S_{5,5,5,3}]=\frac{S_{5,5,5}}{125}[1-3b_0]\quad\ldots\quad(\zeta)$$

Now $$\frac{16}{1000}S_{5,5,5,4,2}-\frac{24}{1000}S_{5,5,5,5}=\frac{1}{125}S_{5,5,5}[2S_{4,2}-3S_5]$$

$$=\frac{S_{5,5,5}}{125}(-u_{-2}+u_{-1}+u_0+u_1-u_2)$$

$$=\frac{S_{5,5,5}}{125}[1-(b_{-1}+b_0+b_1)]\quad\ldots\ldots\quad(\eta)$$

In equations ζ and η, Mr. Woolhouse's method of graduation and Mr. J. A. Higham's second 17-term formula are exhibited in Mr. G. F. Hardy's elegant and distinctive form.

IV. It may be shown that Mr. J. A. Higham's summation formula for Mr. Woolhouse's method of graduation is a particular case of a general summation formula corresponding to the generalized form of Mr. Woolhouse's method.

Taking p curves and following Mr. Woolhouse's method, u_0 is determined as the arithmetical mean of the results obtained by putting $z=-\frac{p-1}{2}$, $-\frac{p-3}{2}$, . . . successively up to $\frac{p-1}{2}$ in the expression

$$\frac{z(z+p)}{2p^2}u_{z-p}+\frac{p^2-z^2}{p^2}u_z+\frac{z(z-p)}{2p^2}u_{z+p}$$

The pair of results corresponding to $z=-\frac{p-1}{2}+r$ and $z=\frac{p-1}{2}-r$ will give

$$-\frac{p^2-(2r+1)^2}{8p^2}\left(u_{-\frac{3p-2r-1}{2}}+u_{\frac{3p-2r-1}{2}}\right)$$

$$+\frac{p^2-\left(\frac{p-2r-1}{2}\right)^2}{p^2}\left(u_{-\frac{p-2r-1}{2}}+u_{\frac{p-2r-1}{2}}\right)$$

$$+\frac{(p-2r-1)(3p-2r-1)}{8p^2}\left(u_{-\frac{p+2r+1}{2}}+u_{\frac{p+2r+1}{2}}\right)$$

The final result for the graduated value of u_0 may therefore be written

$$\frac{u_0}{p}+\Sigma_{r=0}^{r=\frac{p-3}{2}}\left[-\frac{p^2-(2r+1)^2}{8p^3}\left(u_{-\frac{3p-2r-1}{2}}+u_{\frac{3p-2r-1}{2}}\right)\right.$$

$$+\frac{p^2-\left(\frac{p-2r-1}{2}\right)^2}{p^3}\left(u_{-\frac{p-2r-1}{2}}+u_{\frac{p-2r-1}{2}}\right)$$

$$\left.+\frac{(p-2r-1)(3p-2r-1)}{8p^3}\left(u_{-\frac{p+2r+1}{2}}+u_{\frac{p+2r+1}{2}}\right)\right]$$

or, putting $p-2r-1=2s$,

$$\frac{u_0}{p}+\Sigma_{s=1}^{s=\frac{p-1}{2}}\left[\frac{-s(p-s)}{2p^3}\left(u_{-(p+s)}+u_{(p+s)}\right)+\frac{p^2-s^2}{p^3}\left(u_{-s}+u_s\right)\right.$$

$$\left.+\frac{s(p+s)}{2p^3}\left(u_{-(p-s)}+u_{(p-s)}\right)\right]$$

which may be written

$$\frac{u_0}{p}+\Sigma_{s=1}^{s=\frac{p-1}{2}}\frac{p^2-s^2}{p^3}(u_{-s}+u_s)+\Sigma_{s=\frac{p+1}{2}}^{s=\frac{3p-1}{2}}\frac{(p-s)(2p-s)}{2p^3}(u_{-s}+u_s)$$

In this form the result agrees with that given by Mr. C. D. Higham (*J.I.A.*, xxvi, 50).

Now, consider the following scheme:

Term of Original Series	Assumed Value	Summation in p's	2nd Summation in p's	3rd Summation in p's
$u_{-\frac{3(p-1)}{2}}$	0	0	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{-r}	0	0	0	$\frac{(3p-2r-1)(3p-2r+1)}{2.4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{-\overline{p-1}}$	0	0	1	$\frac{(p+1)(p+3)}{2.4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{-\frac{p-1}{2}}$	0	1	$\frac{p+1}{2}$	$\frac{p(p+1)}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{-s}	0	1	$p-s$	$\frac{3p^2+1}{4}-s^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_0	1	1	p	$\frac{3p^2+1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{\frac{3(p-1)}{2}}$	0	0	0	1

This scheme, which is constructed on the very convenient plan adopted by Mr. J. A. Higham in his papers in the 25th and 31st volumes of the *Journal* (*J.I.A.*, xxxi, 319), exhibits the coefficients with which the term u_0 enters into the S corresponding to each term of the original series from $u_{-\frac{3(p-1)}{2}}$ to $u_{\frac{3(p-1)}{2}}$. Further, since the term u_{-r} enters into the S corresponding to u_0 with precisely the same coefficient as that with which u_0 enters into the S corresponding to u_{-r}, the coefficients in column (5) of the scheme are the coefficients of the various terms in the S corresponding to u_0; that is:

$$S_{ppp}=\Big[u_{-\frac{3p-3}{2}}+3u_{-\frac{3p-5}{2}}+\ldots+\frac{(3p-2r-1)(3p-2r+1)}{2.4}u_{-r}$$

$$+\ldots+\frac{p(p+1)}{2}u_{-\frac{p-1}{2}}+\ldots.+\left(\frac{3p^2+1}{4}-s^2\right)u_{-s}$$

$$+\ldots+\frac{3p^2+1}{4}u_0+\ldots\ldots+u_{\frac{3p-3}{2}}\Big]$$

the coefficient of u_{-r} being the general type of the coefficients up to that of $u_{-\frac{p-1}{2}}$, and the coefficient of u_{-s} being the general type of the coefficients of the remaining terms up to the central term. Now operate on the right-hand side of the above expression with $\frac{3p^2+5}{8}-\frac{p^2-1}{8}S_3$, that is, substitute $\left[\frac{p^2+3}{4}u-\frac{p^2-1}{8}(u_{-1}+u_1)\right]$ for u.

The coefficient of u_0 becomes

$$\frac{3p^2+1}{4}\cdot\frac{p^2+3}{4}-\frac{p^2-1}{4}\cdot\frac{3p^2-3}{4},\ \text{which}=p^2.$$

The coefficient of u_{-s} becomes

$$\frac{p^2+3}{4}\left(\frac{3p^2+1}{4}-s^2\right)-\frac{p^2-1}{4}\left(\frac{3p^2+1}{4}-s^2-1\right),\ \text{which}=p^2-s^2$$

The coefficient of u_{-r} becomes

$$\frac{p^2+3}{4}\cdot\frac{(3p-2r-1)(3p-2r+1}{8}-\frac{p^2-1}{8}\left[\frac{(3p-2r-3)(3p-2r-1)}{8}\right.$$

$$\left.+\frac{(3p-2r+1)(3p-2r+3)}{8}\right]$$

which reduces to $\frac{1}{2}(p-r)(2p-r)$.

$$\text{Hence}\quad\frac{3p^2+5}{8p^3}S_{ppp}-\frac{p^2-1}{8p^3}S_{ppp.3}$$

$$=\frac{u_0}{p}+\Sigma_{s=1}^{s=\frac{p-1}{2}}\frac{p^2-s^2}{p^3}(u_{-s}+u_s)+\Sigma_{s=\frac{p-1}{2}}^{s=\frac{3p-1}{2}}\frac{(p-s)(p-2s)}{2p^3}(u_{-s}+u_s)$$

which, it will be seen, is the expression already obtained for the average value of u_0 derived from p third difference curves. In other words $\frac{3p^2+5}{8p^3}S_{ppp}-\frac{p^2-1}{8p^3}S_{ppp.3}$ is the summation formula

corresponding to the generalized form of Mr. Woolhouse's method. The error to fifth differences will be found from (β) to be $-\dfrac{(17p^2+7)(p^2-1)}{1920}(\Delta^4-2\Delta^5)u_0$.

Putting $p=5$, the formula gives $\dfrac{1}{125}\left[10S_{5.5.5}-3S_{5.5.5.3}\right]$, Mr. Higham's summation formula for Mr. Woolhouse's graduated u_0, with an error of $-\dfrac{27}{5}(\Delta^4-2\Delta^5)u_0$.

CORRESPONDENCE.

ON THE GENERAL EXPRESSION FOR THE FORCE OF MORTALITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The general expression for the value of the force of mortality given by you, but not demonstrated, in a foot-note to Mr. Sheppard's communication in the last number of the *Journal* (*J.I.A.*, xxxii, 295), appears to me to be of sufficient importance to call for a formal proof, and I therefore venture to communicate the following.

Let the value of l_x be expressed, by means of Lagrange's interpolation formula, in terms of $l_{-n} \ldots . l_0 \ldots . l_{+n}$; we shall have

$$l_x = \sum_{k=-n}^{k=+n}\left[l_k \frac{\overline{x-(-n)} \ldots\ldots\ldots \overline{x-(+n)}, \text{ omitting } x-k}{\overline{k-(-n)} \ldots\ldots\ldots \overline{k-(+n)}, \text{ omitting } k-k}\right]$$

The general term, whether k be $+$, 0 or $-$, may be put into the form

$$l_k \frac{\dfrac{\overline{x+n} \ldots\ldots \overline{x-n}}{x-k}}{\underline{|n+k}\ \underline{|n-k}}(-1)^{n-k}$$

whence

$$\frac{l_0-l_x}{x} = \frac{l_0}{x} - \sum_{k=-n}^{k=+n}\left[l_k \frac{\dfrac{\overline{x+n} \ldots\ldots \overline{x-n},}{x\ \overline{x-k}}}{\underline{|n+k}\ \underline{|n-k}}(-1)^{n-k}\right]$$

$$= \frac{l_0}{x} + \sum_{k=-n}^{k=+n}\left[l_k \frac{\dfrac{\overline{x+n} \ldots\ldots \overline{x-n},}{x\ \overline{x-k}}}{\underline{|n+k}\ \underline{|n-k}}(-1)^{n-k+1}\right] \quad \ldots\ldots \quad (a)$$

Put $x=0$; then $\frac{l_0-l_x}{x}=-\frac{d}{d_x}l_0=l_0\times\mu_0$, and for all values of k, except zero, the general term in the expression (a) becomes

$$l_k\frac{\frac{\underline{|n}\times\underline{|n}\times(-1)^n}{-k}}{\underline{|n+k}\;\underline{|n-k}}(-1)^{n-k+1}=\frac{l_k}{k}\cdot\frac{(\underline{|n})^2}{\underline{|n+k}\;\underline{|n-k}}(-1)^{2n-k}$$

The term involving l_0 becomes, by adding the two coefficients which appear in (a),

$$\frac{l_0}{\underline{|n}\;\underline{|n}}\times Lt_{x=0}\left[\frac{\overline{x+n}\;\ldots\ldots\;\overline{x+1}\cdot\overline{x-1}\;\ldots\ldots\;\overline{x-n}(-1)^{n+1}+(\underline{|n})^2}{x}\right]$$

which takes the form $\frac{0}{0}$, but it may be shown that when $x=0$ the term vanishes (*). We therefore have

$$\mu_0=\Sigma\frac{l_k}{kl_0}\cdot\frac{(\underline{|n})^2}{\underline{|n+k}\;\underline{|n-k}}(-1)^{2n-k},\text{ omitting } l_0.$$

Now we have, since any even power of (-1) is equal to unity

$$\frac{(-1)^{2n-k}}{k}=-\frac{(-1)^{2n--k}}{-k}$$

i.e., the coefficient of l_{+k} is the same as that of l_{-k} with the sign changed; hence the terms may be arranged in pairs. It is also evident that the signs are alternately + and −, and we shall have finally

$$\mu_0=\frac{l_{-1}-l_{+1}}{l_0}\;\frac{(\underline{|n})^2}{\underline{|n+1}\;\underline{|n-1}}-\frac{l_{-2}-l_{+2}}{2l_0}\;\frac{(\underline{|n})^2}{\underline{|n+2}\;\underline{|n-2}}+\;.\;.\;.\;.\;.\qquad(\beta)$$

(*) $\overline{x+n}\;\ldots\;\overline{x+1}\;\overline{x-1}\;\ldots\;\overline{x-n}\;(-1)^{n-1}$

$=\overline{x^2-n^2}\;\overline{x^2-\overline{n-1}^2}\;\ldots\;\overline{x^2-1^2}\;(-1)^{n-1}$

$=\overline{n^2-x^2}\;\overline{\overline{n-1}^2-x^2}\;\ldots\ldots\;\overline{1^2-x^2}\times(-1)$

$=-[(\underline{|n})^2+\text{ terms involving }x^2,\text{ \&c.}]$.

Thus, when $x=0$ the fraction of which we require the limiting value is in the form $\frac{\text{Terms involving }x^2,\text{ \&c.}}{x}$, and the limiting value is therefore zero.

In Mr. Sheppard's formulæ the numerical coefficient of $\dfrac{l_{-k}-l_{+k}}{l_0}$ is given in the form

$$\frac{\dfrac{\underline{|n}}{\underline{|k}\ \underline{|n-k}}}{(k+n)\ldots\ldots(n+1)}\underline{|k-1} = \frac{\dfrac{\underline{|n}}{\underline{|k}\ \underline{|n-k}}}{\dfrac{\underline{|n+k}}{\underline{|n}}}\underline{|k-1} = \frac{(\underline{|n})^2}{\underline{|n+k}\ \underline{|n-k}}\,\frac{1}{k}$$

as given above.

The formula having been obtained by the use of $2n+1$ values of l will be correct to $(2n)$th differences.

I am, Sir,

Your obedient servant,

GEORGE J. LIDSTONE.

Bartholomew Lane, E.C.,
February 1896.

ERRATA.

International Congress of Actuaries.

In the list of names of the Permanent Committee, given on page 237 of the present volume, the following corrections should be made, namely:

For Wolterbuck read Wolterbeek.
,, Coja ,, Toja.
,, Paul ,, Penl.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Books and Forms to be used in Scheduling the Particulars of the Risks of a Life Assurance Company under its Assurance and Annuity Contracts for Periodical or Interim Valuations, Distribution of Surplus, and for Investigation of the Rates of Mortality, Surrender, and Lapse. By James Chatham, F.I.A., F.F.A., *Secretary and Actuary of the Scottish Life Assurance Company.* (*Messenger Prize Essay,* 1895.)

[Read before the Institute, 24 February 1896.]

THE method of valuation to be adopted in any office, and the books and forms to be used in scheduling the particulars necessary for it, must be regulated to a great extent by the requirements of "The Life Assurance Companies Act, 1870"; and it will be convenient here to give a brief summary of those requirements so far as they relate to the subject of this essay.

Under the Summary and Valuation appended to the Fifth Schedule of that Act, the particulars required for each class of policy, are the number of policies, the sums assured and bonuses, and the office yearly premiums; also the values of the sums assured and bonuses, and of the office yearly premiums, and the net liability. The *net* yearly premiums are only required, if ascertained; and their value, if computed. The extra premiums payable are required separately for with and without profit policies,

the term "extra premium" being defined in a foot note as "the charge for any risk not provided for in the minimum contract premium"; and the particulars of the re-assurances are required *in toto* only.

Under the Sixth Schedule of the Act, the sums assured and bonuses on policies for the whole term of life, also the premiums receivable annually after deducting the abatements made by the application of bonuses (distinguishing ordinary from extra), are required for each age for with and without profit policies separately. For policies other than for the whole term of life, the sums assured and bonuses and the yearly premiums (distinguishing ordinary from extra), are required for each class only, for with and without profit policies separately; also the total amount of premiums received from the commencement upon all policies in force in each class. The amount of immediate annuities for each age is also required; and for annuities other than immediate, the amount of annuities payable, the amount of premiums receivable annually, and the amount of consideration money received, in respect of each class, also the total amount of premiums received from the commencement upon all deferred annuities.

While a prominent position must be accorded to the requirements of "The Life Assurance Companies Act, 1870", there are considerations of great importance to insurance offices, which must be allowed their due weight in determining the method by which those requirements are to be complied with. Hitherto, offices, as a rule, have been content to accept the surplus at an investigation, without analyzing it, to ascertain the sources from which it has come, and how much each source has contributed to it. So long as the offices were in their youth, or even middle age, and a comparatively light mortality was experienced along with a good rate of interest, there was perhaps little or no necessity for such an analysis; as the other principal source of profit—namely, loading—is one which can be kept constantly in view. But circumstances are changed. Many of the offices now in existence were established at the end of the last or early in the present century, and are now no longer young; and the Act above mentioned has had the effect of restricting the number of new offices. New business has not the same effect in keeping down the rate of mortality as formerly, in consequence of the smaller proportion it now bears to the old business upon the books. Notwithstanding the great advances which medical science has made, it is open to question whether the lives now admitted

by insurance offices are as good as those which entered in the early part of the present century, and it seems pretty certain that at any rate the incidence of the mortality has changed. The outlets for capital during recent years have become less adequate to the supply; and the consequence has been a fall in the rate of interest all over the world, so that there is less chance now than formerly of substantial profit being made upon investments. Competition among offices is also keener, and the cost of keeping up the supply of new business shows a decided tendency to increase. In view of all these conditions, it is becoming more and more important for every office to analyze its sources of profit, and to see how these are to be maintained, and how the profit when realized ought to be distributed.

But it is not sufficient to know merely how the surplus as a whole has arisen. It is important to know whether sufficient rates are being charged for the various classes of business transacted. Fire and accident insurance offices have found it necessary to investigate different classes of risks, in order to see whether remunerative rates were being charged; and the same holds good in other professions, such as banking. In those cases the effect of unremunerative rates is immediate and apparent, and the remedy can be at once applied. It is not so in life assurance. There the effects may only be felt after a long series of years, when, perhaps, it is too late to remedy them. It is, therefore, even more important for the manager of a life insurance office than for the manager of a fire or accident insurance office, or of a bank, to see that the business is being conducted on remunerative terms, and that the various classes are contributing their proper quotas to the surplus.

Not only, however, is it necessary to see that remunerative rates are being charged, but also that substantial justice is being done as between different classes of policyholders in the distribution of surplus. For example, the question how much is contributed by whole life and endowment assurance policies respectively, has been much debated. To meet the popular taste, new features are being introduced, and special classes are thus becoming more numerous; so that it is increasingly desirable to investigate the profit-yielding qualities of different classes of risks.

Another consideration which must be kept in view is the labour attending an investigation into the affairs of an assurance company; and in this respect the system at present in vogue in most offices is far from satisfactory. It is usual for the work

connected with it to be commenced when the quinquennial or other period is drawing to a close; and the extra work it involves throws a considerable strain upon a staff which is adapted to overtake the ordinary work only. The consequence is that an investigation is frequently more costly than the ordinary work of the office, and not so well done; and when the final results come into view, there is little time left for making any desirable change, such as a reduction in the rate of interest or the adoption of a more stringent mortality table. Neither are the books arranged in such a way as to adapt themselves readily to an investigation made at any other time than the particular date in view.

As contrasted with the prevailing system, the valuation forms and schedules of an office, to be complete, should fulfil the following conditions, namely:

(1) They should be adapted to meet the requirements of the "Life Assurance Companies Act, 1870."
(2) They should admit of an analysis of the surplus at an investigation, into its component parts.
(3) They should exhibit the profit or loss on each separate class of business.
(4) These results should be obtainable with a minimum of labour, and in a simple manner, at the end of one, three, five, or any other number of years that may be desired.

Let us now examine the various attempts that have been made at different times to solve the problem we are discussing, and see how far they comply with the conditions we have laid down as essential.

Actual and Expected Deaths. From the earliest times insurance offices have been in the habit of comparing the actual deaths, with those expected according to the mortality table employed in the valuation; and that practice has been continued by a large majority of the offices down to the present day. Mr. Low, in an interesting paper "On the method of comparing the expected with the actual "experience of a life insurance company, as regards the number "of deaths and amount of claims" (*J.I.A.*, xviii, 195), discusses the method usually adopted. The lives or sums assured (or both) on the books at the beginning of the year, are arranged according to age attained: and if the new entrants and withdrawals during the year are uniformly spread over it, one-half of the former is added, and one-half the latter deducted. If the new entrants

are not uniformly spread over the year, a suitable adjustment is made. This gives the number exposed to risk of death during the year; and the expected deaths or claims (or both) are calculated, the same function q being used in both cases for the usual classes of policies. Assuming the new entrants and withdrawals to be uniformly spread over the year, the formula for the expected deaths will therefore be:

$$q(s+\tfrac{1}{2}n-\tfrac{1}{2}w)=q[s+\tfrac{1}{2}(n-w)]$$

where s represents the number of survivors at any age on the books at the beginning of the year, n the new entrants, and w the withdrawals, during the year; q having its ordinary significance. We may write the formula for the expected claims

$$q(\text{S}+\tfrac{1}{2}n\text{S}-\tfrac{1}{2}w\text{S})$$

where S represents the sums assured. The special classes require to be dealt with individually as a rule; and sometimes a different function is necessary, according as the expected deaths or the expected claims, are required; for instance, in the case of last survivor policies.

Subsequently, Mr. Ryan, in a paper "On the method of estimating expected deaths and expected claims" (*J.I.A.*, xxvi, 249), suggested that the lives at risk should be separated into two classes: (1) those that have been under observation for five years and upwards, and (2) those under observation for shorter periods; the expected deaths among the former body being calculated by the rate of mortality according to the $\text{H}^{\text{M}5}$ table, and those among the latter by a rate of mortality applicable solely to the first five years of assurance. He further suggested that, while the expected deaths should be derived from the ordinary rates of mortality, the expected claims should be obtained in a similar way, by employing rates of loss based upon amounts insured.

True Death Strain—Higham's Method.

"But when it is remembered," says Mr. C. D. Higham, in an able paper "On the true measure of the death strain on the funds of a life assurance society" (*J.I.A.*, xx, 153), "that for every policy on the books of "an office some reserve value, of greater or less amount, is "in hand; it is apparent that no system of comparison of actual "and expected mortality can be correct which does not take "into account the extent to which the claims are, and should be, "met by the reserve values held against them. The actual cost to "the office of the death of an assured life, is not the sum assured,

"but the difference between that amount and the reserve value "retained to meet such a payment; and it is this balance which "should be the measure in any attempt to ascertain the strain of "the mortality on the funds". Mr Higham's arrangement of the facts to obtain the desired result is as follows: For each age there is required the sum assured and reserve value of

(1) The policies in force at the end of the year.
(2) The death claims which occurred during the year.
(3) A proportion of the assurances terminated during the year otherwise than by death.
(4) A similar proportion of the new business.

With regard to the proportion, he considers one-half sufficiently accurate. The first, second, and third items, are added together, and the last deducted; and this gives the amount exposed to risk during the year and the reserve value at the end of the time. He recommends that the amount exposed to risk and the reserve value be separately multiplied by the probability of dying in a year at the age attained at the commencement of the year, rather than the difference between the two; as the additional insight gained is well worth the slight additional labour involved; but adds that it will probably be found in most cases that the calculations will be sufficiently accurate if made in quinquennial groups of ages with such a value of q as best meets the case. The difference between the two products thus found, shows the strain provided for according to the table; and the difference between the actual claims and their reserve values, shows the actual strain. If the actual strain is less than the expected, there is a profit to the extent of the difference; if greater, there is a loss to that extent. His formula is therefore:

$$\begin{aligned} & q(S_{+1}+dS+\tfrac{1}{2}wS-\tfrac{1}{2}nS) \\ & -qV_{+1}(\qquad \text{do.} \qquad) \\ & =q(S+\tfrac{1}{2}nS-\tfrac{1}{2}wS) \\ & -qV_{+1}(\quad \text{do.} \quad) \end{aligned}$$

which it will be observed is the same as the formula previously given, except that the reserve value is introduced.* Mr. Higham states that his method applies to cases where annual valuations are made.

* In writing the reserve value at the end of the year, the symbol V_{+1} has been used throughout for the sake of clearness, instead of the more correct symbol ${}_{+1}V$.

Ryan's Method. Mr. Ryan, in a paper "On a means of calculating the expected death strain in a life office" (*J.I.A.*, xxx, 196), says, that the reason why an examination of the true effect of the claims experience of an office upon its funds is not so common a thing as might be expected, is the great difficulty of applying the principle explained by Mr. Higham upon the lines laid down by him. He says there is no doubt that the conditions imposed by Mr. Higham can only be complied with at a great expenditure of time and labour, more especially in the case of an office transacting a large new business with its attendant waste from surrenders and lapses; and he shows how the result can be obtained with less labour. His formula for whole-life policies with uniform premiums, according to the notation used above, is

$$q[\mathrm{S}-\mathrm{SA}_{+1}+\mathrm{SP}(1+a_{+1})].$$

He states that the necessity for valuing the sums assured and bonuses, and net premiums, at an age one year older, prevents the ordinary valuation schedules from being used; but that the re-arrangement is simple enough, and would not take very long to work out. No account, however, is taken of the new entrants or the withdrawals of the year; but some of the difficulties will, he points out, disappear if the usual units of value are discarded in favour of continuous values. The value of a policy may be assumed to grow by momently instead of annual increments, and the formula then becomes

$$\mu(\mathrm{S}-\mathrm{S}\bar{\mathrm{A}}+\mathrm{SP}\bar{a});$$

or, if continuous values are not employed in the valuation,

$$\mu\left[\mathrm{S}-\mathrm{SA}\left(1+\frac{i}{2}\right)+\mathrm{SP}\left(\frac{1}{2}+a\right)\right].$$

The advantages of this formula, he points out, are that the quantities and their values are brought into strict correlation as to age, thus permitting direct use of the results already obtained in the usual valuation schedules; and that many troublesome and complicated adjustments in respect of new entrants and withdrawals, are got rid of; for no such disturbances need be considered to take place at the instant of time to which the formula strictly refers. A rough estimate is made for the expectations under special policies. He adds that the results give, with quite sufficient accuracy, in the case of an office not subject to rapid expansion or contraction, the mean expected claims and strain, of

the years immediately preceding and immediately succeeding the date of investigation.

Burridge's Method. Mr. Burridge, in the discussion which followed the reading of the paper, pointed out that in the latter case the new bonuses would require to be added, so that a separate investigation must be made for them. The plan he advocated was to make a test valuation at quinquennial ages, with the assistance of the class-book at the end of the first year of a new period. Then, by applying the appropriate q at each group of ages, he obtained the required results for the past year; and, by applying q one year older to the same figures, the figures for the coming year. The process could be repeated at the end of the third year of the quinquennium, so as to obtain the figures for the third and fourth years. Thus, by means of a test valuation, repeated twice in a quinquennium in addition to the ordinary valuation, he obtained the figures relating to each year of the period. The necessary corrections must, of course, be applied in respect of new business and of discontinued policies; but it would be found in practice that the whole operation did not involve much trouble.

Prospective and Retrospective Methods. All the methods which have been propounded for solving the problem we are dealing with, have been based upon the well-known formula:

$$\begin{aligned}(V+P)(1+i) &= q+pV_{+1}\\ &= q(1-V_{+1})+V_{+1};\end{aligned}$$

and those we have been considering employ the first term only of the right hand side of the equation, $q(1-V_{+1})$, the value of q being taken from the mortality table used in the valuation. This gives the expected strain, and the actual is found by deducting the reserves from the claims. The result shows the profit (or loss) from mortality. But this result may be obtained in another way. The formula may be written:

$$(V+P)(1+i)-q-pV_{+1}$$

or preferably,

$$l(V+P)(1+i)-d-l_{+1}V_{+1};$$

and if for d, the expected claims, we substitute the actual claims (assuming the sum assured in each case to be 1), and make the necessary reserve for the survivors only, we obtain the profit from mortality. It is obvious that in these circumstances the office premiums would be known; and if the net premiums are deducted, the loading is found; and if from the result the expenses are deducted, the profit from loading is also found. Again: the

interest actually received would be known; and if the interest added to the net premiums and to the reserve is deducted, the profit from interest is found. This method, known as the retrospective, is, I imagine, a very old one, and has often been used in practice—more particularly in annuity funds—to ascertain the profit or loss from mortality. For instance, it was used by Mr. A. J. Finlaison in his interesting report on the Manchester Corporation Waterworks Life Annuities (*J.I.A.*, xxiv, 377). I think, however, that in insurance offices it has been recognized more in theory than in practice. So far as I am aware, it was first used in direct connection with the subject on hand by Mr. Meikle, in a remarkable paper, or rather series of papers, which he read before the Actuarial Society of Edinburgh during the session 1862-3, and afterwards published in a pamphlet entitled "An analysis of the profits of life assurance." He there sets forth the principles and arguments on which his theories for the determination and distribution of profit proceed, describing fully the prospective and retrospective methods, which names were, I believe, first given by him. He says of the methods: "There may "be nothing new about them: both formulas may be very well "known, but we may learn something new by their comparison. "We gain information from the one method of calculation, which "the other does not afford; and by combining their different "features we may be enabled to give a solution to any problem "which may partake partly of the one method and partly of the "other." He gives a number of interesting tables showing the progress of a fund on various assumptions.

In the discussion which followed the reading of Mr. Ryan's last mentioned paper, Mr. G. F. Hardy brought forward the advantages of the retrospective method in combination with the prospective. He says that, excluding for the moment the question of surrenders and lapses—which in some offices are very small, and in all offices may be allowed for without much difficulty—we have the following relation: To the reserve at the commencement of the year, increased by a year's interest, add the difference between the net premiums and the claims of the year, increased by half a year's interest, and deduct the reserve at the end of the year; the result gives the exact profit or loss from mortality during the year. The method may be easily extended to a quinquennium by accumulating the total reserves at the commencement of the quinquennium for the five years, and adding the accumulated difference between the net premiums and the

claims, deducting from the resulting total the reserve at the close of the period. Put into symbols this becomes:

$$V_0(1+i)^5+\left\{\begin{array}{l}(P_1-d_1)(1+i)^4\\+(P_2-d_2)(1+i)^3\\+(P_3-d_3)(1+i)^2\\+(P_4-d_4)(1+i)\\+(P_5-d_5)\end{array}\right\}\times\left(1+\frac{i}{2}\right)-V_5$$

where V_0, V_5 are the initial and final reserves, and P, d, &c., the total net premiums and claims of each year. We then have a numerical result which is the profit from mortality during the period, including profits from lapses and surrenders. If the profit from withdrawals is to be eliminated, we must record the reserves upon lapsed and surrendered policies, as well as the reserves on claims. It is, however, unnecessary to calculate the reserves on claims, this being one of the advantages of the method. In this way the lengthy calculation necessary for obtaining the expected death strain is avoided.

Searle's Method. Lastly, Mr. Searle, during the same discussion, and afterwards in a paper "On the progress of profit in a life assurance fund" (*J.I.A.*, xxx, 493), submitted an ingenious plan of determining the reserve at the end of a year from the reserve at the beginning. He showed that the value of V_{+1} can be found in terms of V. We have

$$(V+P)(1+i)=V_{+1}+q(S-V_{+1})$$

Putting $(V+P)(1+i)=V'$, we get

$$V'=V_{-1}+q(S-V_{+1})$$

whence $$V_{+1}=\frac{V'-qS}{1-q}$$

and since $$p+q=1$$

$$V_{+1}=\frac{p+q}{p}V'-\frac{q}{p}S$$
$$=V'-k(S-V')$$

k being the "mortality multiplier" $=\dfrac{q}{p}$.

It will be seen that by means of Mr. Searle's method, not only can the expected death strain be calculated, but also the liability at the end of the year; and this again can be used to calculate the expected death strain in the next year, and the liability at the end of it; and so on from year to year. His method of procedure is briefly as follows: The class lists having been posted up for a financial year, and the totals for each year of birth in the

separate classes ascertained, these totals for whole-life, limited payment, endowment assurance, &c., policies are summarized and added together, the result being the total for each year of birth on single lives on the books at the beginning of the year. The new policies, claims, surrenders, bonus surrenders, dropped policies, alterations, &c., are then entered in separate schedules arranged according to year of birth; and each of these again is summarized, so as to obtain the totals for each year of birth. The reserve values are inserted in these summaries; and as he assumes that all policies coming under observation do so at the end of the financial year of entry, and that all cancelled policies, except claims, pass from observation at the beginning of the financial year, the odd time of the year of entry and exit being dealt with separately, the reserve values are calculated as at the end of the year for new business, claims, &c.; and as at the beginning of the year for surrenders, bonus surrenders, dropped policies, alterations, &c., annuities-due being used. A digest of these reserve values, arranged according to year of birth, is then made up; and the totals for each year of birth carried to the corresponding totals of the class lists. The materials are thus prepared for the application of the "mortality multipliers".

As new policies were assumed to come under observation at the end of the financial year of entry, and cancelled policies to pass from observation at the beginning of that year, an adjustment is necessary. In both entries and exits, Mr. Searle says, it is convenient to start with V and work forward to V_{+1}. For new policies the values calculated were those of V, the values for the beginning of the financial year of entry—generally negative values—and these were accumulated by the formula into V_{+1} values for the end of the year of entry, at which point they fall into the general observation. For surrenders and other cancelments the values were equally V, but these part from the general observation at the beginning of the year of exit, and run to V_{+1} for the sake of odd time only. From the values at the beginning and end of the year thus obtained, the proportionate contribution to mortality was ascertained.

Although it is possible to classify whole-life and endowment assurance policies together, Mr. Searle thinks it will be more convenient to keep them separate. Special classes, however, are not so easily dealt with; for instance, in the case of last survivor assurances, each policy is practically treated as if it were three separate policies.

An important factor in Mr. Searle's method is the premium account. The office premiums require to be divided into net premiums, loading, and extras. Further, the net premiums require to be divided into premiums on policies in force throughout the year, and premiums attaching to the years of entry and exit. Further still, the net premiums of entry and exit require to be subdivided under their several categories, as premiums paid on new policies, claims, surrenders, lapsed policies, or otherwise. Furthermore, the net premiums on policies in force throughout the year, as ascertained from the accounts, require to agree with the net premiums in force throughout the year, as ascertained from the valuation class lists. To fulfil all these functions, he devised a special form of premium book.

It will be seen that Mr. Searle's method, though ingenious, is somewhat elaborate. In the discussion which followed the reading of the paper, Mr. Ryan said that he had lately carried through such an operation as that which the author proposed; and anything more laborious and troublesome, it had not often been his lot to undertake. And, after all, the work of the periodical valuation must be gone through. Mr. Searle says: "The method is not proposed as a substitute for the periodical "valuation. If it were, it would be liable to criticism, because "any errors (in taking out the reserve values of the claims for "instance) would be compounded and carried on for ever." But, even if it were not liable to that criticism, it could not be put forward as a substitute, because the "Life Assurance Companies Act, 1870", requires the values of the sums assured and bonuses, and of the office yearly premiums, to be given separately; and Mr. Searle's method does not do this. I look upon this as a fatal objection to his method, or indeed to any method that has the same defect. The Board of Trade may indeed alter the forms contained in the Schedules to the Act, for the purpose of adapting them to the circumstances of any office, or of better carrying into effect the objects of the Act; but I do not suppose that such a vital change as would be rendered necessary by the adoption of any such method, would be generally sanctioned. In a few cases returns in this form have been accepted, but they have been made chiefly by foreign offices under special conditions. Meanwhile, any books and forms, to be satisfactory and likely to be generally adopted in practice, must be capable of yielding readily, not only the profit from various sources, but also the particulars necessary for the requirements of the "Life Assurance Companies Act".

Summary of Methods. I have now described the various methods which have been proposed at different times for ascertaining the profit from mortality, &c. It is somewhat difficult to tabulate succinctly their advantages and disadvantages; but it would be of great assistance in determining their relative merits, and I have, therefore, endeavoured in the following table to show them, at any rate in broad outline:

Description of Work and Results	Higham	Ryan	Searle	Prospective and Retrospective
WORK:				
Values of Sums Assured and Premiums . .	Yes	Yes	No	Yes
Reserve Values of—				
New Business separately	Yes	No	Yes	No
Death Claims. . .	Yes	Yes	Yes	No
Surrenders and Lapses .	Yes	No	Yes	Yes*
Premium Account . .	No	No	Yes	Yes
Mortality Multiplier . .	Yes	Yes	Yes	No
Period when Valuation or Reserves required . .	Yearly	Biennially	Yearly	Optional
RESULTS:				
Profit from Mortality for .	1 year	2 years	1 year	Any no. of years
„ Interest . .	No	No	Yes	Yes
„ Loading . .	No	No	Yes	Yes
„ Surrenders and Lapses. .	Yes	No	Yes	Yes
Requirements of "Life Assurance Companies Act" .	Yes	Yes	No	Yes

* This would not be necessary for surrenders when the value allowed is based upon the reserve made.

An inspection of the above table shows, I think, that the combination of the prospective and retrospective methods of valuation is superior to all the others, whether we regard the amount of labour involved, or the results yielded. I have therefore adopted it; and I will now proceed to describe the form in which I propose to apply it.

WHOLE-LIFE POLICIES.

I think the simplest plan of showing the method is to take examples, and I will therefore suppose that the following whole-life policies with uniform premiums were issued during 1894 on the lives of males born between 1st July 1859 and 30 June 1860, it being assumed that the valuation is to be made as at 31 December; also that the office premiums for £100 assured are based upon the H^M 3 per-cent Table, with a loading of $\frac{1}{19}$th and a constant of 2*s.* 6*d.* I will further suppose that half-yearly and quarterly premiums are formed by adding 5 and $7\frac{1}{2}$ per-cent respectively to the yearly premium, and that the policies are effected just before the ages given.

Date of Entry	Policy No.	Age at Entry	Sum Assured	Premium		How Payable	Date of Renewal	First Payment	
				Ordinary	Extra			Ordinary	Extra
1894				£ *s.* *d.*	£ *s.* *d.*			£ *s.* *d.*	£ *s.* *d.*
Feb. 27	25,000	34	500	11 16 3		Yly.	Feb. 27	11 16 3	
May 9	25,100	$34\frac{1}{2}$	1,000	12 11 8	5 5 0	H-yly.	May 9 Nov. 9	12 11 8	5 5 0
Aug. 21	25,200	35	300	3 16 9		Do.	Aug. 21 Feb. 21	3 16 9	
Nov. 25	25,500	35	1,500	9 16 3	6 1 0	Qly.	Nov. 25 Feb. 25 May 25 Aug. 25	9 16 3	6 1 0
Dec. 29	25,900	35	500	12 3 4		Yly.	June 29	6 7 11	

* The dates of renewal have been written thus, as it is easier to see how many payments of premium fall due in the year of issue.

These examples do not represent the proportions in which half-yearly and quarterly premiums occur in practice, the great majority of policies being usually effected by yearly premiums; but they are typical cases, and have been selected with the view of showing how the method applies to policies effected by premiums payable once or oftener in the year. They all explain themselves with the exception, perhaps, of the last, where the premium is for six months only at first, and yearly thereafter. Cards are now frequently written in offices for policies as they are issued; and they are convenient for the present purpose. They can, as a rule, be readily adapted to it, if they do not already contain the necessary particulars. Cards for re-assurances may be conveniently put along with the principal card in a pocket envelope of corresponding size with printed particulars on the face of it. If this is done, no change can be made on an original policy without the re-assurances being prominently brought under notice, which is an advantage. The re-assurance cards should

be printed in coloured ink to distinguish them from the ordinary, and the particulars written on them should also be in coloured ink.

Valuation Register. I will now give the form of book in which I propose the cards should be entered, and this I call my "Valuation Register". The figures on the first line relate to the policies in force at the end of the previous year, and give the particulars necessary for a valuation. The figures below the first summation will be explained presently.

VALUATION REGISTER.—WHOLE-LIFE POLICIES.

Born between 1 *July* 1859 *and* 30 *June* 1860.

Date	Policy No.	Lives: Male	Lives: Female	Sum Assured	Office Yearly Premium: Ordinary: Due	Office Yearly Premium: Ordinary: Not Due	Office Yearly Premium: Extra: Due	Office Yearly Premium: Extra: Not Due
				£	£ s. d.	£ s. d.	£ s. d.	£ s. d.
B.F.	150	110	10	122,500	2,616 5 3		44 10 0	
1894								
Feb. 27	25,000	1		500	11 16 3			
May 9	25,100	(*a*) 0		1,000	25 3 4		10 10 0	
Aug. 21	25,200	1		300	3 16 9	3 16 9		
Nov. 25	25,500	1		1,500	9 16 3	29 8 9	6 1 0	18 3 0
Dec. 29	25,900	1		500	6 7 11	5 15 5		
	155	114	10	126,300	2,673 5 9	39 0 11	61 1 0	18 3 0
					21 15 5		5 5 0	(*b*)
					2,651 10 4		55 16 0	(*c*)
Dec. 31	3	2		2,000	24 7 11		5 5 0	(*d*)
					2,627 2 5		50 11 0	
					39 0 11		18 3 0	(*e*)
	152	112	10	124,300	2,666 3 4		68 14 0	(*f*)

(*a*) This life is assumed to have been insured before, and to be already included in the 110 brought forward as above.
(*b*) "Unpaid" on terminated policies to be deducted. See Claim Valuation Register, page 409.
(*c*) Premiums paid during year.
(*d*) Terminated during year with "paid" to be deducted. See Claim Valuation Register, page 409.
(*e*) Premiums "not due" to be added.
(*f*) In force at end of year.

These columns explain themselves, with the exception perhaps of those headed "Due" and "Not due". The "Due" column is intended for the whole or any portion of the yearly premium or premiums, falling due in the financial year of entry; and the "Not due" column is intended for the balance. In any case where more than a year's premium is paid at the outset, it is intended that the excess should be entered in a separate list under the class to which it relates, and a year's premium only

entered in the Valuation Register. The number of these cases is, as a rule, so small that it seems unnecessary to provide space for them in the book, as it would require two more columns to allow for ordinary and extra premiums. In any office, however, where such cases are numerous, it would be an advantage to have columns for them, as these would serve for one or two other cases. By means of the column headed "Lives" the rate of mortality may be ascertained. A number of pages will be required for each year of birth, and on the last of them the re-assurances should be entered. I propose that these should be in coloured ink to prevent any possibility of mistake. It will be observed that the second column has been used to show the number of policies.

When the cards have been entered, it will be convenient to keep them in numerical order; and a reference to the book and page where the particulars are inserted, should be put on each card. This is not absolutely essential in the case of ordinary classes, or even for special classes; but it affords a ready means of sorting the cards according to the order in which they appear in the registers, should that at any time be desirable for checking or other purposes.

Checks on Valuation Register.

There is a complete check upon this part of the Valuation Register. The total new business must agree with the total in the register of policies, which, I suppose, is kept in every office. The addition of the totals of the "Due" and "Not due" columns gives, of course, the premium income from the new business; but in a large number of offices it will be possible to reconcile the totals of these columns separately, as the proportion of the premium income applicable to the year of issue is known. The register of policies can easily be arranged so as to give this information. These checks can be applied at any time during the year, or they may be delayed until the end of the year, when they will involve no extra work.

Claim Valuation Register.

I will now give the form of book in which to enter the claims, surrenders, lapses, &c., which I call my "Claim Valuation Register". I will suppose that Policy No. 25,000 has become a claim during the year with which we are dealing, and that No. 25,100 has lapsed by non-payment of the second premium; also that No. 15,000, effected (say) 10 years ago at age 25 for £500 at a yearly premium of £9. 3*s*. 9*d*., has been surrendered just before payment of the premium falling due in 1894. The following are the entries which would require to be made in the book:

Claim Valuation Register.—Whole-Life Policies.

Born between 1 *July* 1859 *and* 30 *June* 1860.

Date	Policy No.	Lives				Sum Assured			Office Yearly Premium				Surrender Value	Reserve
		Male		Female					Ordinary		Extra			
		Deaths	Not Deaths	Deaths	Not Deaths	Claims	Surrenders	Lapses	Paid	Unpaid	Paid	Unpaid		
1894						£			£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	
June 10	25,000	1				500			11 16 3					
Dec. 10	25,100		0					1,000	12 11 8	12 11 8	5 5 0	5 5 0		
Dec. 24	15,000		1				500			9 3 9			41 11 0	56
	3	1	1			500	500	1,000	21 7 11	21 15 5	5 5 0	5 5 0	41 11 0	56
						500								
		1				1,000								
		2				2,000								

I think these entries explain themselves, and consequently it will be unnecessary to give this form of book for other classes. All that is required is, that the particulars appearing in the Valuation Register shall be written off. Any re-assurances should be entered at the end of the year of birth in the same manner as in the Valuation Register.

Checks on Claim Valuation Register. A summary is now made of the totals of the Claim Valuation Register for this class, as well as for other classes; and the grand total of the claims and of the surrender values should, of course, agree with the amounts appearing in the revenue account furnished to the Board of Trade. The grand totals of the number of policies, the lives, the surrenders, the lapses, and the office yearly premiums, should agree with the records which are generally kept in offices in one shape or another, so as to arrive at the number of policies, lives, sums assured, and premium income, at the end of any year. If this is not already obtainable from the books of the office, it would be advisable to institute such a record, so as to have an efficient check upon the Claim Valuation Register. I have assumed the checks to be applied at the end of the year; but it is obvious that they may be applied at any time during the year, if that is considered desirable. I may mention that it may be more convenient in some offices to have separate books for the claims, surrenders, and lapses, as they may be made to form part of the book-keeping system of the office and be useful for other purposes. One advantage of having separate books would be that the premium income applicable to each section would be known. The systems in vogue in different offices are so varied that no general rule can be laid down.

Particulars for Valuation. The totals of the Claim Valuation Register are carried to the Valuation Register; the totals of the "Unpaid" columns in the former being first of all subtracted from the "Due" columns in the latter. This subtraction gives the amount of premiums paid during the year. The remaining totals are then subtracted, and the totals of the "Not due" columns in the Valuation Register added to the "Due" columns, as shown in the example, the result being the particulars for valuation at the end of the year corresponding to the particulars with which we started at the beginning of it.

There is one point to which attention should be directed. If a policy is effected by *quarterly* premiums in the second or third quarter of a calendar year, and it lapses or becomes a claim *in the same year,* care will have to be taken to write off in the

"Unpaid" column only what is unpaid *for that year*. The difference required to make up the full year's premium must be inserted in the "Unpaid" column also, but in square brackets, or with some other distinguishing mark, and be deducted from the total of the "Not due" column before it is added to the total of the "Due" column. The same remarks apply to a policy effected by half-yearly premiums in the second half, or by quarterly premiums in the fourth quarter, becoming a claim in that year.

Checks on Particulars for Valuation. Complete checks can be applied to this part of the work also. The totals of the premiums paid during the year should be taken down, and the grand total (including any proportions in the separate list previously mentioned) should, of course, agree with the amount in the revenue account. The grand totals of the "Due" and "Not due" columns, along with the grand totals of the "Paid" and "Unpaid" columns in the Claim Valuation Register already obtained, are used to check the additions and subtractions necessary to obtain the premiums for valuation. If the grand total of the "Unpaid" column is deducted from the grand total of the "Due" column, it is obvious the result will be the premiums paid during the year appearing in the Valuation Register; and if from this the grand total of the "Paid" column is deducted, and the grand total of the "Not due" column added, the result should agree with the grand total of the office yearly premiums for valuation and the extra yearly premiums. The other particulars are checked by means of the records already referred to. In a number of offices, separate accounts are kept for various sections or classes: and this will of course facilitate the application of the checks. These checks involve no extra labour, as the particulars for valuation are used for that purpose.

Valuation. The particulars for valuation are now taken down from the Valuation Register on sheets of the following form:

Assumed Date of Birth 31 Dec.	Age at Valn.	No. of Policies	Sums Assured	Office Yearly Premiums						Value of Sums Assured	Value of Ordinary Office Yearly Premiums
				Ordinary			Extra				
			£	£	s.	d.	£			£	£
...	..	..	...	..			..			..	..
1859	35	152	124,300	2,666	3	4	68	14	0	53,921	50,445
...	...	..	...	...			...			...	...
		1,957	2,130,780	66,127	5	10	357	5	0	1,174,532	820,339

Any re-assurances should be entered in coloured ink underneath the totals for the year to which they belong. The advantage of this is that they can be readily subtracted before the values are calculated, except of course in the year in which the returns have to be furnished to the Board of Trade, when it is necessary to ascertain the values of the re-assurances separately. It may, however, be desirable to ascertain the profit or loss from the re-assurances, in which case also they will require to be valued separately.

In making the valuation as at 31 December, I have assumed that the H^M 3 per-cent Table is used in the calculations, and that the sums assured are paid, on the average, two months after death; also that the premiums fall due and are received, on the average, eight months after date of valuation, and that the surrenders, &c., take place after the same interval. I do not think it necessary that any decimals should be retained in the values. The old-fashioned style of calculating the values to three places of decimals is, I am glad to see, fast disappearing. Assuming that 20 per-cent of the premiums are by half-yearly payments and 5 per-cent by quarterly, the reserve is arrived at as follows, it being borne in mind that the manner in which the premiums are loaded is known.

	£	£
Value of Sums Assured		1,174,532
„ Office Premiums	820,339	
„ Loading	82,546	
„ Net Premiums		737,793
		£436,739
Add two-thirds of Extra Premiums		238
Reserve		£436,977

Profit from Mortality, &c.

Profit from Mortality.

Assuming that the reserve at the beginning of the year (including the value of any new bonus declared), arrived at in the same way, was £418,163, the method of determining the profit from mortality, &c., in this class, is as follows:

	£	£	£
Reserve at beginning of Year			418,163
Add Interest at 3 per-cent			12,545
			£430,708
Reserve Values of Policies transferred during year, say		254	
Add Interest at 3 per-cent for four months		3	
			257
Net Premiums received		59,565	
Extra ,, ,, ...		332	
		£59,897	
Claims paid, say	41,527		
Reserve Values of Surrenders, &c., say	7,763		
		49,290	
		£10,607	
Add Interest at 3 per-cent for four months ...		106	
			10,713
			£441,678
Reserve at end of year as above			436,977
Profit from Mortality			£4,701

It will be observed that the class could, if desired, be readily broken up into sections, and the profit in each of these ascertained. This method of splitting the whole-life class into sections would be very useful in some offices, as showing how the bonus allotted to them compares with their contribution to the surplus.

It will also be observed that, although a combination of the prospective and retrospective methods has been used to determine the profit from mortality, the arrangement of the Valuation Register would admit of Mr. Highams' method being applied.

It is not necessary that the valuation be made annually. It may be made at the end of two, three, five, or any other number of years, the premiums, claims, &c., for each year being set out as above, and accumulated at interest. The result would, of course, be the profit or loss from mortality during the period between the two valuations.

Net Premiums. In the above example I have supposed the valuation to be made by the same table as the office premiums are calculated from, also that the proportion of the premium income applicable to half-yearly and quarterly policies is known; and that, consequently, the net premiums can be ascertained with very little trouble. It would be an easy matter to determine exactly the proportion of the premium income applicable to half-yearly and quarterly policies, by the introduction of two columns, in which

to enter the excess of the premiums payable in a year over the yearly premium; but even if this were done, it is only in rare cases that the table used in the valuation is the same as that upon which the premiums are based. Nor is it at all necessary for the method proposed that it should be so. All that is required is that the net premiums be calculated and inserted along with the office premiums in a similar manner. There would not be upon them, however, the same efficient checks that there are upon the office premiums, since there is no independent account kept of the net premiums received during the year. I fear that such an account would involve too much labour. What I would propose instead is that the loading be also calculated and inserted; and then there is an efficient check, since the sum of the net premiums and the loading must be equal to the office premiums. Altogether, therefore, four more columns would be required.

But I would ask whether this minute accuracy is really necessary, and whether the labour of calculating the net premiums cannot be dispensed with? The net premiums are required by the "Life Assurance Companies Act" only if they have been computed; and I think it is quite practicable to find them in most cases with sufficient accuracy from the office premiums. The great bulk of the insurances in an office are effected between ages 30 and 40; and if from that decennium the average constant and percentage necessary to be deducted from the office premiums so as to produce the net premiums, are determined, it will be found, I think, that the result of using such an average deduction is sufficiently accurate for practical purposes. I applied this method to an actual case, using ages 30 and 40 only, that is to say, I found the percentage, k, and constant, c, from the following equations:

$$\varpi_{30}(1+k)+c=\mathrm{P}_{30}$$

$$\varpi_{40}(1+k)+c=\mathrm{P}_{40}$$

where P is the office premium; and after making the necessary allowance for the additional loading on half-yearly and quarterly policies, the estimated net premiums were £113,142, the correct amount being £113,098, or a difference of only £44. I do not put forward the particular method used as being applicable to every case. No doubt in some offices it would be better to introduce other ages so as to obtain good results. Reductions in the rate of interest used in the valuation are now very general, and the tendency is for the net premiums to approximate to the office premiums, and in some cases even to exceed them. The

net premium valued should in no case exceed, and ought to fall short of the office premium so as to allow for expenses, &c.; and an easy way of doing this is to value the office premium less a loading. It would appear, therefore, that the office premium will occupy a more prominent place in valuations than hitherto.

It seems to me in any case quite unnecessary, when the net premiums have been calculated at one rate of interest, to recalculate them at another rate. For instance, if the net premiums have been calculated at, say, H^M 3 per-cent, then by means of them the net premiums at H^M $2\frac{1}{2}$ or $3\frac{1}{2}$ per-cent could be determined very accurately in the manner described above.

Bonus Column. Hitherto I have assumed that the policies have been effected without profits, but the forms already given can easily be adapted for policies with profits. In the Valuation Register only one additional column, headed "Bonus", would be required; and in the Claim Valuation Register the corresponding column would be subdivided into claims, surrenders, lapses, and interim. As this holds good for other classes with profits, it will not be necessary to refer to it again. In some cases a "Not vested" column may be necessary. In general, however, provision is made at an investigation for all bonus as if it vested immediately; and it is convenient to treat it so in any subsequent valuation, entering as profit the reserve for bonus not vested when the policy lapses or is surrendered.

Endowment Assurance Policies.

I will now take up the class next in importance, namely, endowment assurances.

Usual Methods. Formerly it was the practice in a number of offices to value each individual policy separately; but the class has increased so rapidly that a tendency has grown up of late to group them in some way or other, so as to reduce the large amount of labour which the older method entailed. If the ascertainment of the rate of mortality be regarded as a *sine qua non*, the endowment assurances must be grouped according to age attained; and as generally a large proportion of them are payable at certain specified ages, a convenient method of arranging them is according to age attained under each specified age. The number of payments remaining to be made under policies grouped in this way does not usually vary by more than unity, and therefore the average number for each age attained can readily be

ascertained and used in the valuation. Although a certain advantage is obtained in this way by grouping all those of the same age attained under each specified age, the method still involves a considerable amount of labour. It is only at certain specified ages that it is worth while grouping according to age attained, and a large number of policies is left, which are generally valued individually. Suppose that an office has 5,000 endowment assurances. Then it would probably be possible to arrange about 80 per-cent of them under, say, four specified ages, each of which would take quite as long as the ordinary whole-life policies to value, taking into account the more elaborate formula necessary; and the remaining 20 per-cent would probably have to be valued individually. Sometimes, however, the policies maturing at intermediate ages are included in the nearest group. I estimate that these 1,000 policies would take at least six times the amount of labour required to value the 4,000 policies grouped under the four specified ages, so that the endowment assurances would require about ten times the amount of labour required to value the whole-life class. The question therefore arises, is an exact determination of the rate of mortality worth all this labour? I think not. In the first place, it would only be a "mixed" rate of mortality, which would vary according to the amount of new business transacted and the subsequent surrenders and lapses; and in the second place, there does not seem to be the same necessity for ascertaining the rate of mortality when the profit from mortality is known. I think, therefore, that we must search for some other method of valuation for this class, and in doing so it is desirable to look upon the amount of labour required to value the ordinary whole-life policies as a maximum. I do not think that any method for valuing any but a few special classes can be considered satisfactory, if it involves more labour than is required for that class.

New Methods. Some years ago attention was directed to the small effect which, within certain limits, age has upon short-period annuities of the same duration. In December 1889, a paper "On bonuses on endowment assurance policies" (*J.I.A.*, xxviii, 257) was read before the Institute by the late Mr. A. W. Sunderland, in which he pointed out that the bonus tables for endowment assurance policies having the same period to run, were practically the same; so that it may be said, with reasonable accuracy, that the tables of bonuses are independent of the age at entry, and depend only on the difference between the age at entry and the age at which

the policy matures. In the discussion which followed the reading of the paper, Mr. King said that actuaries would possibly value endowment assurances by discarding the question of age, and by taking into account the endowment term only. Mr. Manly said that he had himself found some time ago, that the term the policy had to run was more important in the valuation than the ages; and as he had to make an annual valuation, he had adopted a plan of classifying endowment policies according to the year of maturity. He found the average age by multiplying the sum assured by the age and dividing the sum by the total sum assured, and then valued the whole of the policies in each class as an endowment assurance running for the unexpired term, and he found the results agree very closely with those obtained by valuing each policy separately. Curiously enough Mr. McLauchlan in the same month and year read before the Actuarial Society of Edinburgh (Transactions, Vol. 2, page 333) a paper, "On some formulas for use in life office valuations", in which he referred to the subject. He said: "Throughout a considerable period of life "the values, at adjacent ages, of such temporary annuities as are "commonly required, differ very slightly from each other. ". . . . For instance, the difference between an annuity for "10 years at age 10, calculated at H^M $3\frac{1}{2}$ per-cent, and a similar "annuity at age 40, is only ·318. We see from this, that when "the number of payments of the annuity is 10, endowment "assurance policies on lives aged between 10 and 40 at valuation "may, with sufficient correctness, be included in the same group, "and valued at the common age of say 35. This principle, the "application of which was suggested to me some years ago by "Mr. Meikle, may be used in other cases, with the effect of very "greatly reducing the number of groups in which endowment "assurance policies must be classified for valuation." An example of the valuation of a group of endowment assurance policies, including yearly and half-yearly policies, was also given. The extreme valuation ages in the policies dealt with were 29 and 39, and he used a common valuation age y found from the equation

$$a_{y\overline{n}|} = \tfrac{1}{2}(a_{29\overline{n}|} + a_{39\overline{n}|}).$$

Subsequently Mr. King read a paper before the Institute "On "legislation affecting life assurance companies, more especially "with reference to the 'Life Assurance Companies Acts', 1870 "to 1872, and their amendment" (*J.I.A.*, xxix, 481). In this

he referred to the difficulty of complying with heading No. 6 of the Sixth Schedule, which asks for the total amount of premiums received from the commencement upon all policies under each special class; and suggested that this information be no longer required, but that, instead, such particulars should be given of the endowment assurance policies and of policies by limited payments, whether they were originally of these descriptions or had become such by bonus, as would enable an approximate calculation to be made of the respective reserves. "As regards "endowment assurances, a return might be supplied showing "in columns the sums assured and the reversionary bonuses, "respectively, maturing in each calendar year; and the office "premiums, the net premiums, and the bonus reductions of "premium corresponding thereto, and in another column the "average present age of the lives assured in each yearly group. ". . . . The average present age will be arrived at by "adding together the ages of all the lives assured under policies "maturing in, say, 1892, and dividing by the number of policies. "By experience I can say that a table prepared on this plan "enables a valuation to be checked with very great accuracy, "and with very little trouble. Similarly for limited payment "policies. The sums assured and bonuses should, as in ordinary "cases for the whole term of life, be arranged under present "ages; but, as regards the premiums, the policies should be "grouped as shown above for endowment assurances. In this "way the valuation of policies by limited payments could easily "be checked."

Proposed Method. Taking everything into consideration, I think there is little doubt that the best method of valuing endowment assurance policies, is to group them according to the year of maturity. It is more convenient for reference: it will harmonize in most cases with the financial year of the office: and it will show the amount which will have to be met on maturity in any year: and the duration can be obtained very accurately. I do not propose the adoption of this method merely as a check upon the valuation, and I think, therefore, that it will be better to ascertain the average present age as accurately as possible. I have accordingly adopted Mr. Manly's plan, with obvious modifications. The column headed "Lives", therefore, in the form of Valuation Register already given should be omitted, and the following columns added:—

Total Premiums Received		(Assumed year of Birth − 1800) × Sum Assured ÷ 100	Correction for Last Payment
Ordinary	Extra		

There will, of course, be corresponding columns in the Claim Valuation Register. If the number of any policy as it goes off is struck out in coloured ink in the Valuation Register, it is obvious that not only will the aggregate amount of sums assured falling due in the year of maturity be known, but also the individual sums of which it is composed. It seems unnecessary, therefore, to have a separate column in the Claim Valuation Register for sums paid on survivance, as the few claims by death in the year of maturity can easily be allowed for.

The sum of the third of these columns divided by the number of hundreds in the total of the sums assured gives, of course, the average year of birth minus 1800. This does not take bonuses into account in the case of with-profit policies; but the omission will not, I think, materially affect the results. The last column has been introduced for the purpose of ascertaining more accurately the average number of premiums remaining to be paid. Starting with the assumption that the last year's premium is payable in the year before maturity, the column is intended to show the amount by which the actual payments fall short of, or exceed, that assumption. For instance, if in the case of a half-yearly policy, one half-yearly premium only is payable in the year before maturity, then $-\frac{1}{2}$ would be inserted in the column; if, on the other hand, one half-yearly premium is payable in the year of maturity, then $+\frac{1}{2}$ would be inserted in the column. The sum of the column divided by the number of policies gives the average, which should be deducted from the integral number of payments found on the assumption that a full year's premium is payable in every case in the year before maturity. Strictly speaking, the amount of the premium should be taken into account; but the number of cases in which it will be necessary to use the column is small, and as the payment is deferred, its value will also be small.*

* Since writing the above, it has occurred to me that a better plan would be to insert in the "correction for last payment" column the *amount* by which the premium payable in the last year falls short of, or exceeds, a full year's premium. The "office yearly premiums" would then be valued as if an integral number of payments were to be received, and the value of the amount of the "correction" deducted, treating it as if it were an endowment. This method would necessitate two multiplications to obtain the value of the premiums, but I think there is little doubt it would be both simpler and more accurate.

Test Valuation.

To test the accuracy of the method of valuation here recommended, I made a valuation by the H^M 3 per-cent table, as at 31 December, of 1,985 policies payable at age 60 or death, assuring £522,118 with bonus additions of £44,710, the office annual premiums being £19,791. 15*s.* 5*d.*, and the corresponding net premiums £15,922. 12*s.* The dates of birth and of maturity were assumed to be uniformly spread over the year, both of these assumptions being found by trial to be exceedingly near the truth. The cards were first of all arranged according to age attained in the manner I have already described, and a valuation was made; and then they were re-arranged according to year of maturity, and another valuation was made in the way here proposed. In the second case the average date of birth was found in the manner previously explained correct to one decimal place, and the difference between the year of valuation and the average date of birth thus found gave the age to be used. As the sums assured were assumed to be uniformly spread over the year of maturity, the duration for them was taken as the difference between the years of maturity and valuation less half a year. The total of the "Correction for Last Payment" column for each year of maturity was divided by the number of policies, and the result, correct to one decimal place, subtracted from the difference between the years of maturity and valuation less 1. This gave the duration for the premiums, the age used for them being the same as for the sums assured. The following are the results:

	Value of		Reserve
	Sums Assured	Net Premiums	
First Method . .	341,643	205,443	136,200
Second Method .	341,801	205,417	136,384
Difference . .	158	26	184

It will be seen that the two methods give practically the same results, the difference being only ·13 per-cent of the reserve by the second method; and as the first method ought to give very accurate results, and is, I believe, adopted in a number of offices, we may conclude that the method proposed is sufficiently accurate

for practical purposes.* In valuing by the second method, I should mention that in calculating the values of the sums assured and bonuses, which were assumed to be uniformly spread over

* I have applied the proposed method of valuation to a number of endowment assurance policies with tontine bonuses maturing at various intervals, taken indiscriminately, in a different office. Altogether, there were 112 policies assuring £23,750, and the sums assured were used in the calculation in the manner described above. The following are the results:

Year of Maturity	Value of Sums Assured by		Value of Net Premiums by	
	Exact Method	Proposed Method	Exact Method	Proposed Method
1910	2,155	2,151	1,462	1,482
1916	3,163	3,169	2,330	2,337
1917	2,620	2,610	1,948	1,928
1918	2,861	2,848	2,264	2,283
1919	1,679	1,676	1,280	1,280
	12,478	12,454	9,284	9,310

It will be seen that the values agree on the whole very well, both individually and collectively. The total value of the sums assured by the proposed method is less than the total by the exact method by £24 (·19 per-cent), and the total value of the net premiums more by £26 (·28 per-cent). I should mention that, according to the arrangements in force in the particular office referred to, an integral number of payments were to be received in every case at date of valuation. The reserve by the proposed method is £3,144, and by the exact method £3,194, so that the reserve, excluding bonuses, is too little by £50 (·16 per-cent). The bonuses on these policies were, from various causes, not sufficient to admit of a satisfactory test being applied, and others, therefore, were taken with the following results:

Year of Maturity	Bonuses	Value of Bonuses by	
		Exact Method	Proposed Method
1911	167	82	83
1912	369	156	166
1913	290	129	128
1914	247	105	106
1915	198	76	77
1916	176	66	67
	1,447	614	627

The difference between the totals of the values by the exact and proposed methods is £13 (2·12 per-cent), which is due almost entirely to one year—1912—where there were two or three lives considerably older than the average (see remarks on page 426). The above percentages agree on the whole very well with those given by Mr. Barnes, *J.I.A.*, xxxii, 319.

the year of maturity, the mean of A_{xn} and $A_{x\overline{n+1}}$ was used for $A_{xn+\frac{1}{2}}$. Had the values of $A_{x\overline{n+\frac{1}{2}}}$ been found correct to second or third differences, the error would not have been so great as it is. In general, it will not be necessary to take the average age to one decimal place: the nearest integral age will be sufficient. In those offices where the policies are payable at the end of a given number of years (not a specified age), and a larger amount of business is transacted towards the close of the year, the assurances will not mature in the middle of the year, but nearer the end of it.

Here then we have a method by which, if a suitable table of values has been prepared, the ordinary endowment assurances can be valued as quickly as the ordinary whole-life class. The total premiums received from the commencement, which frequently take several months to calculate, are no longer troublesome. The premiums received during the year are carried to the "Total Premiums Received" column, along with any sums which have been entered in the separate list previously referred to, and the sum of the "Total Premiums Received" column in the Claim Valuation Register deducted. In this way the column is carried on from year to year, and thus in a very short time we obtain the total premiums received from the commencement on policies remaining on the books. These remarks apply to new policies; but all policies existing at the time the Valuation Register is written up, require of course to have set against them the total premiums received from the commencement. The result is often looked upon as of no value whatever, but I have found it useful for checking the liability—for instance, the ratio between the with and without profit classes does not, as a rule, vary to any very great extent. The plan here recommended may not be suitable in every case on account of some peculiarity, such as a special method of allotting bonuses; but it is an easy matter to test it, and, if necessary, to modify it.

Special Policies. But even when the ordinary endowment assurances have been grouped in the way described above, there still remains a large number of policies which cause a considerable amount of trouble in the valuation—I refer to endowment assurances by limited payments, double or half endowment assurances under which double or half the sum assured at death is payable on attaining the specified age, and so on. Now, the form given above is adapted for all these. In the case of limited payments, the sum assured can be inserted in

the year of maturity in which it is payable, while the premium may, to be consistent, be inserted in the year of maturity after that in which it ceases. It will be necessary to put one of the numbers in square brackets or to use some other distinguishing mark, to prevent the policy from being reckoned twice, and this can most conveniently be done when entering the premium. As there is no sum assured along with the premium, the assumed year of birth column must not be filled in, otherwise an incorrect age would be obtained when valuing the policies in the year of maturity in which the premium has been inserted. When the policy is written off either through death, surrender, or lapse, it will of course have to be written off in two places; and when the premium ceases, it will be convenient to carry the total premiums received to the year of maturity in which the sum assured is inserted. Strictly speaking, a column should be provided for the loading being received during the whole currency of the policy; but the interval which elapses between the date of the last payment of premium and the maturity of the policy is generally short, and it does not seem necessary to have this column, especially if the number and amount of such policies are small. This plan prevents the endowment assurances being broken up into sections, and the profit in each section ascertained separately, as the value of the premiums might be in one section, and the corresponding value of the sum assured in another; but this is not a material objection, since policies of all ages at entry and all past durations are grouped under the same year of maturity.

In the case of double and half endowment assurances, it is only necessary to have one more column, subdivided so as to show the amount on survivance to be added or subtracted. Leaving out of account policies effected on the half premium plan, which will be dealt with afterwards, I have found it possible to group *all* the endowment assurances in a large office numbering about 5,000, in the manner described above, with the exception of 11 policies. These would require to be dealt with in a class by themselves, and valued individually, and the Valuation Register should, of course, contain the necessary particulars to enable this to be done.

In cases where a policy matures, it will generally be necessary to make two entries in the Claim Valuation Register,—the first in the year before maturity to write off the premium, otherwise it would be included in the office yearly premiums for valuation, and the second in the year of maturity to write off the sum assured.

LIMITED PAYMENT POLICIES.

Methods.

The class next in importance is that of limited payment policies, that is, whole-life policies in which the premiums are limited to a specified number of payments. Since the premiums are not payable during the whole currency of the policy, this class cannot be grouped in the same way as the whole-life or endowment assurances, and it is, I believe, usual to value each policy separately. This, of course, involves an amount of labour quite out of the question in an annual valuation, unless the number is very small; for, as I have already pointed out, the calculation of the value of each policy takes practically as much time as the calculation of all the whole-life policies, numbering probably hundreds, grouped under one year of birth. In some offices limited payment policies are transferred to the whole-life class when the premiums have ceased; but in others it is not practicable to do this, on account of the method of distribution of surplus being different. If there were then, say 500 limited payment policies, the valuation of these would take at least six times the amount of labour involved in valuing the whole-life class; and if the number of policies were much larger, it is obvious that the calculation of the reserve values would become a very serious matter indeed. If, however, we value the sums assured and the premiums separately as suggested by Mr. King, the difficulty disappears, and the whole class, however large, can be valued in about the same time as the whole-life class. I have, therefore, adopted his method.

Assuming, as is now usual, that the net liability is ascertained by deducting the value of the office premiums from the value of the sums assured plus the value of the whole-life loading (or a proportion of it), the columns required for registering under each year of birth the particulars necessary to value the sums assured and whole-life loading, would be date, policy number, sum assured, whole-life loading. The columns required for registering under each year of last payment the particulars necessary to value the premiums, would be the same as those for the endowment assurances. The first set of columns would be used to ascertain the number of policies in the class, and it would therefore be better to put the totals in the second set in brackets, so as to prevent them from being used, except for purposes of reconcilement. When the premiums on any policies have been all paid

up, the "Total Premiums Received" should be transferred to the single payment policies or to a separate account, in order that the account for that "year of last payment" may be closed.

The net premiums will require to be valued as well as the office premiums, in order that the profit from mortality and interest may be ascertained. The contribution to expenses will, of course, be the whole-life loading, that loading having been substituted for the difference between the office and net premiums.

Test Valuation. I tested the proposed method of valuation in this case also. As the sums assured are grouped in the same manner as the whole-life class, their values will be accurately ascertained, and I therefore directed attention to the premiums, principally with the view of seeing whether the sum assured might be dispensed with in their case in arriving at the average age for valuation. I took a number of policies—110 in all—on which the premiums ceased during the fifth, tenth, fifteenth, &c., years from date of valuation, and valued the office premiums by the proposed method, using the nearest integral age (1) taking into account the sums assured, and (2) omitting them, the "correction" in each case being calculated correct to two decimal places (see, however, foot note on page 419). I also calculated the exact value of the premiums on each policy, and the following are the results:

Duration	Value of Office Premiums by		
	Exact Method	Proposed Method using Nearest Integral Age	
		With Sums Assured	Without Sums Assured
5	3,682	3,689	3,685
10	3,370	3,411	3,446
15	5,253	5,445	5,601
20	1,569	1,584	1,595
25	1,403	1,430	1,430
30	1,533	1,548	1,535
35	5,436	5,479	5,634
	22,246	22,586	22,926

The agreement between the values when the sums assured are used and the values by the exact method is very close in every

case except one, and I think, therefore, it is better to take the sums assured into account. In the exception referred to—duration 15—the excess is due almost entirely to two policies (the values of whose premiums represented more than one-fourth of the whole) having been effected at unusually old ages, namely 50 and 55, the present ages being 59. If great accuracy is desired, therefore, it would be necessary to treat such policies as special, and value them separately. It will be observed, however, that the values when the sums assured are used, are in every case greater; and as the method of ascertaining the net liability when the whole-life loading is reserved, gives a larger reserve than is necessary when the exact values of the office premiums are used, the reserve by the proposed method will be nearer the correct reserve. I found that if the age for valuation were taken correct to one decimal place instead of the nearest integer, it made practically no difference in the results. The omission of the "correction" made a difference of only £79.*

* I have applied the proposed method of valuation of the premiums to the limited payment policies in a different office, and found the results confirm the previous calculation. The durations were taken indiscriminately; and the net premiums were used because the exact values of them had already been computed. Altogether there were 133 policies, and the results are given in the following table:

Duration	Value of Net Premiums by		
	Exact Method	Proposed Method, using Nearest Integral Age	
		With Sums Assured	Without Sums Assured
13	6,259	6,321	6,353
14	4,776	4,823	4,834
18	3,066	3,098	3,083
19	5,805	5,897	5,933
	19,906	20,139	20,203

It will be observed that here also the values by the proposed method are in every case greater than the exact values, although the difference on the whole is small, being 1·18 per-cent of the exact values, when the sums assured are used, as against 1·53 in the previous calculation. This, of course, is due to the inclusion of lives considerably above the average. After age 45 the annuity-value for the same duration rapidly decreases, and the inclusion of a few lives above that age does not increase the average age of the group sufficiently to make a material difference in the annuity-value. There were ten policies on lives over 45 in the above calculation, and the amount of the error due to them, when the sums assured are used, is £174. I should add that an integral number of payments were to be received in every case at date of valuation.

Short Term Policies.

Strictly speaking the sums assured and office annual premiums should be valued separately, also the net premiums if computed; but as is well known these policies have practically no value, and it is sufficiently accurate, therefore, to reserve a proportion of the annual premium income to meet current claims. This being so, it is unnecessary to calculate the values either of the sums assured or of the premiums. The Board of Trade have frequently accepted returns in this form for this and one or two other classes subsequently dealt with, and there is, therefore, no objection to the plan on this score. I accordingly propose to adopt it. Assuming that the first three policies on page 5 were effected for five years only instead of for the whole of life, also that the premiums are calculated in the same way, the loading, however, being $\frac{1}{4}$th instead of $\frac{1}{10}$th, then the following are the entries required to be made in the Valuation Register:

Valuation Register—Short Term Policies.

Date	Policy No.	Sum Assured	Office Annual Premium — Ordinary — Due			Office Annual Premium — Ordinary — Not Due	Office Annual Premium — Extra — Due			Office Annual Premium — Extra — Not Due	Total Premiums Received — Ordinary			Total Premiums Received — Extra	Year of Termination
		£	£	s.	d.	£ s. d.	£	s.	d.	£ s. d.	£	s.	d.	£ s. d.	
B. F.	94	165,500	2,696	15	4		10	0	0		6,690	4	6	25 0 0	
1894 Feb. 27	25,000	500	6	2	11										1899
May 9	25,100	1,000	13	6	8		5	5	0	5 5 0					1899
Aug. 21	25,200	300	2	0	0	2 0 0									1899
	97	167,300	2,718	4	11	2 0 0	15	5	0	5 5 0	6,690	4	6	25 0 0	
			2	0	0		5	5	0		2,718	4	11	15 5 0	
	97	167,300	2,720	4	11		20	10	0		9,408	9	5	40 5 0	

As we have supposed the premiums to be due, on the average, eight months after date of valuation, the reserve will be 8-12ths of £2,720. 4*s*. 11*d*.=£1,813. The corresponding proportion of the extra premiums will require to be reserved also.

In the case of short-term policies effected by single payments, I propose to deduct from the single payment a percentage to cover expenses, &c., say 5 per-cent, and to divide the balance by the number of years the policy has to run. The sum of the yearly decrements for existing policies thus found would be deducted each year from the reserve for the previous year, and the new office premiums less 5 per-cent added. For new policies the first decrement would, in most cases, be not a full year's, but a proportion only. The same would be the case for policies in their last year. By valuing in this way we can at once obtain a reserve which is sufficiently accurate for practical purposes.

Half-Premium Policies.

This class if valued strictly is very troublesome; and as the lower premium is generally only slightly higher than the short-term premium for a similar number of years, I propose to treat the class during the currency of the low initial premium as if the policies were effected by short-term premiums, reserving in addition the difference between these premiums and the premiums actually payable. Assuming the same three policies to be effected by half premiums for the first five years, the following would be the entries required [see next page]:

The reserve will be obtained as follows:

Reserved difference of premium as at 31 December 1893	1,352
$\frac{8}{12}$ths × 2550·6	1,700
Difference of premium in 1894, 3001·8 − 2548·6 ...	453
$\frac{2}{3}$rds of Extra Premiums, $\frac{2}{3}$ × 25·5	17
Reserve as at 31 December 1894	3,522

The difference of premium in 1894, 453·1, would, of course, be carried to the column "Reserved Difference of Premium" and added to 1351·8, as shown in the statement, and the total thus carried forward from year to year.

The same method may be adopted in the case of endowment assurances, &c., effected by half premiums.

Valuation Register—Half Premium Policies.

Date	Policy No.	Sum Assured	Office Annual Premium: Ordinary: Due	Office Annual Premium: Ordinary: Not Due	Office Annual Premium: Extra: Due	Office Annual Premium: Extra: Not Due	Short Term Office Premium: Due	Short Term Office Premium: Not Due	Total Premiums Received: Ordinary	Total Premiums Received: Extra	Reserved Difference of Premium
		£	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£	£	£ s. d.	£ s. d.	
B. F.	70	156,900	2,977 14 8		15 0 0		2,527·2	...	9,461 14 4	65 0 0	1,351·8
1894 Feb. 27	25,000	500	6 19 2				6·1				
May 9	25,100	1,000	14 16 8		5 5 0	5 5 0	13·3				
Aug. 21	25,200	300	2 5 3	2 5 3			2·0	2·0			
	73	158,700	3,001 15 9	2 5 3	20 5 0	5 5 0	2,518·6	2·0	9,461 14 1	65 0 0	1,351·8
			2 5 3		5 5 0		2·0		3,001 15 9	20 5 0	453·1*
	73	158,700	3,004 1 0		25 10 0		2,550·6		12,463 10 1	85 5 0	1,804·9

* For Explanation see below.

ENDOWMENT POLICIES.

Most of the children's and other endowments are effected with premiums returnable in the event of death before the specified age is attained, and the valuation of them by a more or less exact formula is a very laborious matter. As mortality in these cases is practically eliminated, a practice has grown up of simply accumulating the premiums paid at compound interest, and this will be found to give very accurate results in most cases. I propose, therefore, to adopt it. This class of business is not pushed, and it will be found, I think, that the premiums are very evenly distributed over the year. We may, therefore, assume that all the premiums falling due in the first six months are received at the beginning of the year, and those falling due in the second six months at the end of the year. Taking the same three policies, and supposing them to be now 15 years' endowments effected at such an annual premium (returnable in event of death) that it will, when accumulated at 3 per-cent compound interest, amount at the end of the period to the sum assured then payable, the following will be the entries required:

VALUATION REGISTER—ENDOWMENT POLICIES.

Date	Policy No.	Sum Assured	Office Annual Premium		Office Premiums in first six months		Total Premiums Received	Reserve	Year of Maturity
			Due	Not Due	Due	Not Due			
		£	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£	
B. F.	346	92,816	4,821 8 6		2,337 5 10		22,369 13 10	26,361	
1894 Feb. 27	25,000	500	26 2 1		26 2 1				1909
May 9	25,100	1,000	54 16 8		27 8 4				1909
Aug. 21	25,200	300	8 4 6	8 4 6		8 4 6			1909
	349	94,616	4,910 11 9	8 4 6	2,390 16 3	8 4 6	22,369 13 10	2,391	
								28,752	*
								863	
			8 4 6		8 4 6		4,910 11 9	2,520	
	349	94,616	4,918 16 3		2,399 0 9		27,280 5 7	32,135	

* For Explanation see below.

The reserve will be obtained as follows:

Reserve at beginning of year	£26,361
Premiums received during first six months ..	2,391
Together	28,752
Add interest at 3 per-cent	863
	29,615
Premiums received during second six months: £4,910. 11*s*. 9*d*. – £2,390. 16*s*. 3*d*. ..	2,520
Reserve at end of year	£32,135

This, of course, can be done in the Valuation Register, as shown in the example, and it is, therefore, unnecessary to have any separate statement. If the premiums are not evenly distributed over the year, the six months can be altered to any other period. It will generally be found convenient to accumulate the office premiums; but, if desired, other columns can be introduced so as to allow a margin for expenses, the balance only being accumulated.

When any policy is written off in the corresponding Claim Valuation Register, the reserve will, of course, require to be written off along with it, interest being calculated up to the end of the year.

The number of endowments effected without return of premium is generally very small, and the same method may be adopted in their case.

The same plan may also be adopted, with advantage, in the case of deferred assurances where the risk commences on the life attaining a specified age, and the premiums are returnable in the event of death before that age, and in other similar classes. It will be found, I think, that the reserve obtained in this way is sufficiently close for practical purposes.

Other Classes.

Some of the other classes may be grouped so as to reduce the time occupied in valuing them. For instance, in the case of joint-life policies, two equal years of birth may be found, and the policies grouped in the same way as the whole-life class; but the number will generally be found to be so small as not to make it worth while to group them in this way, and they may be valued individually. Last survivor, contingent, deferred, &c., policies may also be valued individually. The Valuation Register should, of course, contain the necessary particulars to enable this to be

done without referring to any other source. The tables of functions for joint lives—equal ages—given by Mr. King at the end of Part II of *The Institute Text Book*, will be found very useful in special cases. Suppose, for instance, that it is desired to find the value of a policy payable on the death of the last survivor of six lives. As the highest number of lives there given is four, we may find the value of an annuity on the four equal ages equivalent to the youngest four. By entering Orchard's table we obtain at once the assurance-value; and as the net premium is supposed to have been already entered in the Valuation Register, a value of the policy is rapidly obtained which is sufficiently accurate for practical purposes. These tables are very useful even in cases where there are only two lives; for instance, in the case of endowment assurance policies on two joint lives, the value can be more rapidly obtained from them than from the Institute tables. For contingent assurances, x against y, we may use Gray, Smith, and Orchard's assurance and annuity-tables, Carlisle 3 per-cent.

Notwithstanding the small numbers in these classes, and the ease with which they can be valued in the manner indicated above, they would occupy more time than their relative importance would seem to warrant if an annual valuation were made. I think, however, that the plan mentioned by Mr. C. D. Higham (*J.I.A.*, xxx, 533) is a good one, namely, to calculate the values at the beginning and end of the quinquennial or other period, and interpolate for any intervening time, and I would therefore adopt it. Sometimes, however, the circumstances of a policy change during the valuation period—for instance, a temporarily reduced premium may cease, and the full premium again become payable; but it seems hardly necessary to take into account such changes, unless they are important, and materially affect the value of the policy. I would have the yearly increase or decrease arranged so as to give the reserve without difficulty.

In all classes where the policies are valued individually, it will be necessary when the policy becomes a claim, or is otherwise dealt with, not only to write it off in the Claim Valuation Register, but also to mark the entry in the Valuation Register in such a way as to prevent the policy being valued on a subsequent occasion. If the policy goes off the books, the striking out in coloured ink of the assumed years of birth will be sufficient. When alterations are made upon policies in these classes, the best plan

is to write off the policy entirely and write it on again afresh, so as to have the particulars for valuation all together.

Hitherto, in describing the various forms of Valuation Registers, I have only dealt with them from a valuation point of view, but I may mention that they serve other purposes. For instance, they will show that

(1) All premiums terminating in a particular year have been written off.

(2) All endowment assurances maturing in a particular year are either paid or held as outstanding.

(3) Half premium policies are transferred to proper class when higher premiums become payable.

(4) Female extras in the whole of life classes are all taken off at the proper time.

(5) Short-term policies have all been taken off on the termination of the risk.

(6) Deferred policies are all transferred to the proper class when the risk commences.

(7) Endowments are all paid on maturity.

Annuities.

I propose that the method of arriving at the profit or loss from mortality, &c., already explained, should be adopted in this case also; and as it is even simpler here, on account of there being generally no disturbing influences, such as surrenders or lapses, I do not think it is necessary to give examples. Assuming the annuities are payable either half-yearly or quarterly, and the first payments due three months after date of valuation, the following is the form I propose for immediate annuities:

Born in Year 189 .

Date	Annuity No.	Purchase Price	Amount of Annuity.				Correction for		Loading
			Male		Female				
			Due	Not Due	Due	Not Due	Payable Quarterly	Proportion to Death	

The only columns requiring explanations are, I think, the second and third from the end. In the first division of the "Correction" section, is intended to be inserted one-eighth of the amount of the annuity when payable quarterly; and in the second division, the proportion which, on the average, will be paid at death when the annuities are complete, in order that we may have a ready means of valuing the annuities accurately.

In the case of deferred and other annuities effected by annual premiums, columns similar to those already given for insured lives will require to be introduced.

SURRENDERS, LAPSES, TRANSFERS, AND ALTERATIONS.

In the form of Claim Valuation Register, I have, as desired, provided columns for surrenders and lapses, but I do not think it is in all cases an easy matter to determine what is a surrender and what is a lapse. If a policy is surrendered during the days of grace, it would, I presume, be treated as a surrender; but suppose that it lapsed by non-payment of the premium, and that the surrender-value is paid shortly afterwards, how should it be treated at the time of writing off the policy, that is to say, before surrender? If as a lapse, why should the delay of possibly a few days make all the difference in the method of treatment? And what would be the value of the rates of surrender and lapse thus obtained? If the surrenders and lapses are to be kept separate, the only consistent course seems to me to be to treat all policies which have acquired a surrender-value as surrenders, and all those which have not acquired a surrender-value as lapses, and to carry the surrender-values of policies not taken within the days of grace to a suspense account. This would obviate the necessity of making another entry in the Claim Valuation Register when the surrender-value is paid after the policy has lapsed. When the period during which the office is liable to pay the surrender-value, varying generally from one to five years, has elapsed, then credit may be legitimately taken for the surrender-value. I have supposed the surrender-value only to be carried to a suspense account; but if a policy is revived, then the office would be represented as sustaining a loss of the difference between the reserve and the surrender-value, and it is open to question whether it would not be better to carry the whole of the reserve to the suspense account. The balance of the account would, of course, be treated as a liability at the valuation. A number of offices, however, have now what

are called non-forfeiture regulations, under which, if the premium is not paid, the surrender-value is applied to keep the policy in force. If suitable entries are made in the account books, it would not be necessary in such cases to have a suspense account.

There is also a difficulty as to how transfers and alterations should be treated, as they are neither surrenders nor lapses,—at all events in a number of cases. Suppose, for instance, an alteration in the year of birth only is made, then it would be necessary to transfer the policy from one year to another (along with its reserve, if we wish to divide the class into sections at any time). Even if a separate column were reserved for them in the Claim Valuation Register, they would still be reckoned as new business in the Valuation Register, and the most convenient course seems to me to be to enter them in a separate list as well as in the registers.

Reduction of Premium.

Permanent. When the premium has been permanently reduced by application of bonus or by a cash payment, the reduction will be written off in the Claim Valuation Register, and effect will have to be given to the change in the net premium if that is inserted. There are so many ways in which this will have to be done, depending upon the method of valuation and the method of distribution, that I do not think it is practicable to discuss them here. The modification of the form so as to make it applicable to a particular office or class can easily be made. The same remarks apply to classes in which the premium has been reduced by discounting future bonus.

Temporary. There are two ways of dealing with temporary reductions: (1) the reduction may be written off in the Claim Valuation Register, and the exact amount payable left, in which case a note of the reduction would require to be taken for valuation purposes, and the reduction written on again when the full premium became payable; or (2) a note may be taken of the reduction for reconcilement purposes, and no entry made in the Claim Valuation Register. The disadvantage of the first method is that it burdens the books with a number of entries; and as the reduction is not required for reconcilement purposes, it may be more readily overlooked in the valuation. The disadvantage of the second method is that the full premium is, in the case of special classes, carried into "Total Premiums Received"; but this can be overcome by deducting from that column the amount of

the reconcilement necessary to obtain the actual premiums received during the year. On the whole I think the second method is the better, and a similar course should be followed when a premium is commuted for a few years at the outset. It has the advantage of keeping the premium next the sum assured in the case of special policies.

Incidental Payments. Following out the principle laid down in the case of temporary reductions, payments of an incidental nature, such as extras for permission to visit foreign countries, fines for postponement of payment of premiums, &c., should not be inserted on the cards nor in the Valuation Register. No difficulty will be experienced in reconciling the amount of premiums received if payments of that nature are kept in a separate account in the books of the office. Strictly speaking, extra premiums even of an incidental nature should be noted on the cards, and entered in the Valuation Register, so as to be carried to the total premiums received in the case of special classes, in order to make a correct return to the Board of Trade; but the amount is generally so small that it may be ignored. Cases in which such a charge is made are every day becoming rarer.

BONUS BOOK.

In a number of offices the policies in this book are arranged in numerical order for convenience of reference; but this would necessitate the insertion of any new bonus on the cards, and the subsequent sorting of them according to the Valuation Register, in order that the total might be entered in the register for each age or year as the case may be. This would, of course, involve a great deal of trouble, and what I propose is that the policies in the Bonus Book should be arranged in the same way as they are in the Valuation Register. It would then be an easy matter to carry the totals of the new bonus to the Valuation Register at the end of the valuation period. I do not think this method of arranging the policies would interfere much, if at all, with the utility of the book. Before a policyholder can deal with his bonus it is necessary to know the class and the date of birth, and with these particulars the policy could be found readily in the Bonus Book. In any case there are always the cards to fall back upon. In classes where the policies are valued individually, it would be necessary also to insert the bonus in the book where the reserve is calculated.

It will be necessary to note all bonuses surrendered, &c., in this book, and I propose that all cash values should be noted

here also. This will render it unnecessary to make any entries in the Valuation Register until the end of the year when a valuation is to be made. The Bonus Book should have columns corresponding to the number of years in the valuation period in which to make the necessary entries. The systems of distribution are so varied that no general form for this book can be given. In those offices where a simple or a compound bonus is given, the form will be a simple one, while in those using what is known as the "contribution method", it will necessarily be somewhat complicated. For methods of shortening the labour in the latter case, I may refer to the interesting paper by the late Mr. Rothery, "On "a system of bonus distribution considered in relation to the "office premiums and modes of valuation" (*J.I.A.*, xxx, 131), and to an exceedingly ingenious plan devised by Mr. R. P. Hardy, to whom the insurance profession is so much indebted, which is printed as an appendix to the opinion of Messrs. Hendriks, Bailey, and Hardy, upon a case submitted to them on behalf of an insurance society (*J.I.A.*, xxxi, 330); also to the interesting paper by Mr. J. A. Robertson, recently read before the Institute.

Selection.

In the examples given I have assumed that the valuation has been made by a "mixed" table, that is to say, a table in which all the lives of the same age attained are classed together, irrespective of the length of time for which they have been assured. A more correct method would, of course, be to take into account the duration of the assurance, and to use select tables; but I do not suppose that this will ever be done on account of the enormous amount of labour it would entail. A number of offices, however, make an allowance for selection by valuing policies under five years in force by the H^M table, and those over five years in force by the $H^{M(5)}$ table with H^M premiums. The method I have given would apply in this case also. The books would simply require to be duplicated, and at the end of five years from entry, the policies would be carried from the one book to the other along with the necessary reserve. A good approximation, however, could be made for selection in the case of whole-life policies by valuing policies on the lives of persons below age 40 by the H^M table, and on the lives of persons above that age by the $H^{M(5)}$ table. I have made a valuation of a large whole-life class with profits by this method, and found it less than the exact valuation by the H^M and $H^{M(5)}$ tables, but the difference was only ·14942 per-cent of the latter.

Division of Surplus.

"It is difficult", says Mr. Meikle in his papers previously mentioned, "to give a clear and distinct definition of profit, its "nature depends so much upon the mode of its calculation. "In general, however, if an office has assumed certain data, and has "experienced other more favourable data, profit may be said to "be the difference between the actual and the calculated experience, "or the difference between the realized and the expected results, "or profit arises from favourable deviations from the fundamental "data." An illustration will, I think, make it clear. Suppose 89,865 persons, aged 30, insure their lives in an office for £100 payable if death occur within one year: also that the office assumes (1) that the mortality to be experienced will be that represented by the H^M table, according to which 694 deaths will occur during the year; (2) that the claims are to be paid at the end of the year; (3) that the rate of interest to be realized is 3 per-cent; and (4) that the expenses incurred at the beginning of the year will be met by loading the net premium according to that table, ·74978 per-cent, with a percentage of 15 and a constant of 2*s.* 6*d.* Then if the rates of mortality, interest, and expenditure experienced, agree exactly with those assumed, there will be neither profit nor loss, and the transaction will work out as follows:

	£
Premiums received at beginning of year ...	88,719
Deduct expenses	21,340
Balance	67,379
Add 3 per-cent interest	2,021
Fund at end of year	69,400
Claims	69,400

Suppose now that the rate of mortality actually experienced is 25 per-cent less, the rate of interest 1 per-cent more, and the rate of expenditure 50 per-cent less, than the expected, then the transaction will work out as follows:

	£
Premiums received at beginning of year	88,719
Deduct expenses	10,670
Balance	78,049
Add 4 per-cent interest	3,122
Fund at end of year	81,171
Claims...	52,050
Profit	£29,121

Here, there is a profit of £29,121 on the transaction, which is made up as follows:

		£
Profit from mortality, being difference between expected claims, £69,400, and actual claims, £52,050		17,350
Profit from interest, being 1 per-cent on £67,379 ...		674
Profit from loading, being difference between		
assumed expenses	£21,340	
and actual expenses	10,670	
	£10,670	
with 4 per-cent interest	427	
		11,097
Total		£29,121

It may at first sight appear that the profit from interest should be stated as £1,101, the difference between £2,021 the assumed amount in the first statement, and £3,122 the amount actually realized according to the second statement; but a little consideration will show that interest on the difference between the assumed and the actual amount of expenses, is really part of the profit from loading.

We must have some standard wherewith to measure the profit from any source, as the absolute amount is sometimes not a reliable index of the relative productiveness of any source; and we may lay it down as a general rule that policies should participate in the profit according to the measure in which they have contributed to it.

Sources of Profit.

Interest. The standard by which I propose to measure the profit from this source is $\frac{A+B}{2} - \frac{I}{2}$, where A and B represent the fund at the beginning and end of the year respectively, and I represents the interest received during the year, a method due, I believe, to Mr. G. F. Hardy. This profit should be distributed according to the value of the policy, and for practical purposes this may be taken at the beginning of the period during which the profit has arisen.

Loading. The forms of books given enable us to determine the total amount of loading, with interest, received during the period under review, and this is the standard by which the profit from this source should be measured. Each policy should receive its share according to the amount it has contributed.

Mortality. As we have seen, the contribution of each policy to this source is $q(S-V_{+1})$, and the standard should therefore, strictly speaking, be $\Sigma q(S-V_{+1})$; but this would necessitate a large amount of calculation, and I propose therefore that an average value should be taken for q. In a number of offices this average value will no doubt vary so little that it may be dispensed with altogether, and the standard taken as the sums assured less the reserve values. For each policy it will probably be found that the contribution during the greater portion of its currency does not vary to any very great extent, and that it may for practical purposes be taken as a constant, and the profit therefore incorporated with the loading.

Surrenders and Lapses. When a policy is surrendered, a deduction is generally made from the reserve value on account of (1) the loss of the contribution to expenses and the cost of obtaining another policyholder, and (2) the loss which is supposed to result from the increased mortality among the remaining lives occasioned by the withdrawal of a more or less healthy life. The first part should be incorporated with the profit from loading, and the second with the profit from mortality. The profit from lapses should be dealt with in the same way, the only difference being that the first part will probably be larger than in the case of surrenders.

Investments. It is usual to keep profit or loss from investments separate—I mean profit or loss on the sale, or writing up or down of any security; and not only is it desirable to do so, but it is also necessary in view of the requirements of the Revenue Account in the Fifth Schedule of the Life Assurance Companies Act, 1870. But when we come to distribute it, I think it should be treated as of the nature of interest, and be incorporated with that item. The late J. S. Mill said: "The "gross profit on capital may be distinguished into three parts, "which are respectively the remuneration for risk, for trouble, and "for the capital itself, and may be called insurance, wages of super-"intendence, and interest." Now, the greater the risk the higher is the rate of interest required. If a decrease, therefore, takes place in the capital value of a security, it is almost invariably due to the element of risk having increased, assuming, of course, that there is no other factor to account for it, such as a repayment at par within a few years of a security standing at a premium. As the element of risk has increased, a higher rate of interest (or, according to Mr. Mill, of "gross profit") is required, and

consequently the capital value is diminished. The converse is the case when an increase in the capital value takes place, although no doubt it is sometimes due to the large amount of money seeking employment. I think the best plan is to create a fund as a reserve, crediting it with any profit and debiting it with any loss, and this fund can be increased or diminished according to the view taken of the securities, but in either case the account operated upon should be "Interest and dividends."

Re-assurances. While it may be desirable to keep the re-assurances separate in order to ascertain whether a profit or loss is being made upon them, I see no reason for doing so when a distribution of surplus is made. What should be ascertained is the profit or loss on the *net* business.

No doubt a modification of some of the above may be desirable in practice, but what direction it should take will depend a great deal upon the particular office that is being dealt with.

Conclusion.

I have now sketched the scheme by which I propose to attain the objects with which we set out, and in doing so I have avoided, as far as possible, the use of technical language. I have submitted what I think is a *practical* scheme, and I have described it somewhat minutely. This may have given rise to the impression that it is a complicated scheme, but it is not so in reality, and I think a glance at the forms proposed for use will remove any such impression. The various adaptations to meet the peculiarities of different classes of risks are such as would have to be made under any system that aims at avoiding the tedious processes of valuing policies singly or in small groups. I have not dealt with the table of mortality or rate of interest to be used in the investigation, as I think these are outside the scope of the present paper; and I have left the question of the particular formulas to be used open as far as possible. It only remains for me to give a few particulars as to the best method of carrying out the scheme in practice.

Assuming the office to have, say 10,000 or 15,000 policies on its books, I think it will be convenient to divide the Valuation Register and the corresponding Claim Valuation Registers into four sections or volumes. The first would contain the whole-life class with profits, and the second the endowment assurances and other classes with profits, while the third and fourth would contain the corresponding classes without profits. It is not at all

necessary that the Claim Valuation Registers should be distinct from the Valuation Registers. One register would do for both, the new business being entered on one page and the claims, &c., on another. Their division, however, has the advantage of enabling the books to be used separately, while their combination has a tendency to make the books unwieldy.

As the scheme is a simple one, it requires very little skilled labour. All the work connected with the registers, and the subsequent abstraction of their totals, does not require any actuarial knowledge, and can be performed by any clerk of average intelligence. There are thorough checks upon every part of the work, with the exception of one or two unimportant columns, and they can be applied at any time. I have no doubt, too, that the cost of the system proposed will be considerably less than the old. The posting of the registers in an office of the size mentioned above would occupy only a small portion of a clerk's time, and, even in the largest offices, it would not occupy all his time. It may be done daily, weekly, monthly, or at such other times as may be found convenient. As the reserves for classes in which the policies are valued individually, such as joint life, last survivor, contingent, &c., are supposed to have been calculated previously,* while the reserves for such classes as short term, half-premium, endowments, &c., are obtained at once, there are only three classes to value, namely, whole-life, endowment assurance, and limited payment. The second and third have really been the great difficulty hitherto; but, by the methods proposed, they will now take no more time than the whole-life class, and this quite irrespective of the number of policies in either of the classes. I think the valuation work should be done in duplicate, and I estimate that it would certainly not occupy two clerks with a good knowledge of actuarial work, assisted by two other clerks, more than a week. It must be borne in mind, too, that the valuation of the largest insurance offices in this country could be made in the same time. It is not, of course, necessary by the system proposed to make the valuation annually, but I do not suppose that the small amount of labour requisite for it would deter any office from undertaking an annual valuation in view of the advantages to be derived from it.

* I estimate from the returns of some of the largest offices that the number of such policies would rarely exceed 300, while in most cases it would be very much less. The large majority of them are joint life and contingent policies, the reserve-values of which are readily calculated.

Discussion.

The President hoped many of the members present would give the meeting the advantage of their opinions upon the interesting subjects so fully discussed by Mr. Chatham. The paper was the Messenger Prize Essay for 1895, the subject of which Mr. Chatham alone had the courage to attack, but with the excellent result disclosed that evening. The terms of reference implied that a concentrated form was required for scheduling the particulars of life assurance and annuity contracts, so that valuations, distributions of surplus, investigations of rates of mortality, surrender, and lapse should all be obtainable from one set of books. Mr. Chatham had proposed a simple form in his paper, one which was more compact than any that he (the President) had seen in actual use in societies that had much diversity in the forms of insurance contracts granted by them, or in which there was not a uniform and simple relation between the premiums that had been charged to their clients from time to time and the net premiums used in a particular valuation. There were many gentlemen present, connected with societies established many years ago, who, he hoped, would express their views on the forms suggested for practical use by the author of the essay.

Mr. R. P. Hardy did not think there could be any difference of opinion as to the highly practical value of the subject in which Mr. Chatham had so distinguished himself. To frame a scheme of Books and Forms that were at once analytical and interlocking, presenting at the same time the maximum of convenience for the needs of to-day, encumbered by the mere minimum of what was indispensable for the emerging wants of the future, was in itself a liberal and highly stimulating education, and one that could not fail to train the faculties to operate with that all-round, steady, but flexible, force which distinguished the experienced man from the mere closet student. He did not wish to occupy the time of the meeting by any detailed discussion of the plans proposed. The names of many of his friends had been specially referred to, and he was sure they would be glad to show how far the results of Mr. Chatham's labours would carry out their intentions.

Mr. C. D. Higham joined in the congratulations to Mr. Chatham upon his success, for the paper must be the result of a great deal of labour. It was specially gratifying to hear Mr. Chatham explain that he had written the essay a good deal from a feeling of obligation laid upon him as a member of the Institute for whom the Institute had done something for which he was bound to respond to the best of his ability. Taking advantage of the frank speech which was allowed in that hall, because the frankest criticism was the truest compliment that could be paid to any paper, he must confess to a little disappointment with the essay before them as showing some want of proportion. The author appeared often to have run off into an excursus upon subjects beyond the scope of the title, and to have elaborated details which were hardly of sufficient importance. He thought details were sometimes too much regarded, and had never forgotten a sentence in a paper read many years ago to the Institute, to the effect that their whole course of business was a system of

compromise between that which was theoretically correct and that which was practically convenient. He would ask the author if the book as designed by him had ever been in actual use. He (Mr. Chatham) had said the book was so simple that any average clerk could work it; but it appeared to him to abound with pitfalls. One cause of the difficulty seemed to him that Mr. Chatham crowded too much into his book, and it was apparent that he realized this to some extent, when he suggested that it should be divided into four volumes. He (Mr. Higham) thought it would be still better to split the book up into several books. There was always a fear when criticising anyone else's books that one might have a bias for one's own, but he thought the best form of valuation class book was one confined to the policy number, the sum assured, the premium payable in the year, and the loading (leaving out the net premium), with the office age for the writings on and the cause of going off on the other side. There was not nearly enough notice taken of the advantage of valuing the loading instead of the net premium, for in that way they could get rid of all difficulty as to special reserve for single premium, and (by increasing the loading) of half-yearly and quarterly premium policies, and would be able to do so also for limited payment policies after a slight alteration of the Act. It was a mistake to mix up the financial aspect of the premiums with the valuation class book. It was far better that the premiums which appeared in the valuation class book under the dates of birth, should be in other books under the months in which they fell due, with also the writings off under the months, and the addition of such books at the end of the year must agree with the totals of the valuation class book. If the policies were valued as yearly policies, they had an asset in half-yearly and quarterly premiums for the current policy year yet to be received; and for these it was very desirable to have a separate book, so that it might be known exactly what the asset was, allowance being made for discount and chance of non-receipt. Mr. Chatham's book was also complicated by questions as to the number of lives, but this could not be helped, in view of the requirements of the essay, though for practical use he (Mr. Higham) thought they need not trouble about the number of *lives* except once in a generation when the Institute called upon them to furnish their cards. The rate of mortality as to *claims* was quite sufficient for their purpose, and the same applied to rates of surrender and lapse. Mr. Chatham referred to his (Mr. Higham's) paper read nearly 20 years ago, and he confessed that the more he used that system in conjunction with Mr. Woolhouse's excellent idea, the more he found it valuable. It might be that he was prejudiced, having spent the best part of his official life in offices not blest with many special policies, but if a rough valuation and a claim estimate could be got out on one side of a sheet of foolscap, there did not seem much more to be desired. He thought gentlemen would find it interesting to take out their estimates at the beginning of the year, and for the actual results the value of each policy separately during the year, as at the previous or next yearly valuation according as the premium had not or had been paid, so that the experience might be compared as the year was running its course. If Mr. Chatham based an assumption on the date on which the claim was *paid*, he must assume also as to

premium receipt the date of *receipt* and not of falling due. The author's estimate of net premiums would need great caution to avoid negative values; while as to his kindly reference to him (Mr. Higham) with regard to special policies, it was undoubtedly a great advantage to value them for three or five years in advance and interpolate, but Mr. Chatham had not mentioned that a book was desirable showing the successive values of the policy on the same page, so that the eye could be run down and see that the figures were about what they ought to be. He himself used a book a little modified from an excellent pattern of Mr. R. P. Hardy's. He thought the author's differentiation of the surrenders and lapses not altogether satisfactory, as it would break down in an office where every policy had a surrender value, and it came back to this, that they must use their own common sense to decide whether a man intended to surrender or whether the policy just ran out.

Mr. H. W. Manly said that, when a clerk of the name of Messenger left a legacy of £200 to the Institute of Actuaries, no doubt he thought he was benefitting a worthy institution, but he had no conception of the use that the money would be put to by the Council in affording an opportunity to the younger members of the profession of displaying their knowledge, and bringing themselves prominently before the notice of their seniors, by establishing a prize out of the accumulated dividends. The paper before them had the distinction of being awarded the Messenger Prize, and he heartily congratulated Mr. Chatham upon this distinction. The author had shown a large and extensive knowledge of his subject, but the paper itself bore some evidence of hasty preparation and completion. There was a boldness in some of the recommendations which was to be commended, and those recommendations deserved their most careful attention. The first observation that would occur to anyone in reading the paper would be that it was deficient in forms. They were all anxious to get their books into such order that they would, as it were, interlock and check each other. Mr. Chatham in his paper assumed rather too much, more particularly where he said—"The grand totals of the number of policies, the lives, the surrenders, the lapses, and the office yearly premiums should agree with the records which are generally kept in offices in one shape or another"; also, "If this is not already obtainable from the books of the office it would be advisable to institute such a record." He had throughout assumed that they had some books in such a form that it was not necessary to teach them how to make a perfect book. In that he had assumed too much. Their books were not quite so perfect as that, and it would have added very much to the value of the paper if the author had given a whole set of forms of books which were necessary for conducting the business of the office, and a classification book for valuation purposes, showing how the information in each would afford a perfect check in the end. He agreed with Mr. Higham in doubting whether the form was really a practical one for use in the office. He should have great hesitation in asking an ordinary clerk to take the care and management of this wonderful book. He would certainly be inclined to throw the whole of its adjustments into a separate sheet at the end rather than to fill up the valuation book by additional and, in his opinion, unnecessary columns

for the unpaid premiums, &c. It would be better, he thought, to accumulate all those items at the end, in one balance sheet, by which the premium account and the valuation account should be reconciled. The author had been exceedingly bold in suggesting the omission of the net premium column in the valuation class book, although he had, as stated, made some exceedingly close approximations. Their premiums had not always been based scientifically upon the H^M, or any other table, and very often they had been arrived at by comparison with the rates charged by other offices. It might be that by averaging the loading upon the premiums from 30 to 40 they would be able to arrive at the correct net premium. Still he thought they would be like Mr. Chatham, inclined to check the assumption by putting the net premiums in. At any rate, until they were satisfied by a check of that kind that the assumption was correct, he did not think they would feel confident in simply taking a percentage off the gross values.

Mr. JAMES CHISHOLM said he was hardly disposed to take the view of the two previous speakers. To his mind the author's point of view in approaching this subject was that of trying to get an analysis of sources of profit, and to work that into a system of office books. He had made some attempts in that way by the application of Mr. Searle's method, and was bound to say at once that it was in the application that the difficulty came in. Of course, there were a great many unexpected circumstances cropping up that had to be dealt with—the searching out of the mistakes that arose in ordinary daily work and checking particulars led to loss of time, and the system required very great care. Where Mr. Searle's method broke down was in the new business. Where it could be applied to a closed series, it was an admirable method, and worked with very little trouble. Asking one of his staff how long it would take to value a certain closed series, he said about a fortnight, and to keep the permanent book, which was necessary, would take about a month of a man's time, so that for that expenditure they got a yearly valuation. His own feeling was that that was a very desirable result to arrive at. When making a quinquennial valuation disclosing so much profit, it was natural to wish to know where it had come from, and they all more or less tried to estimate the source. If they could only feel sure that any system, such as Mr. Searle's, could be constructed so as to get over the difficulty they had to meet with of dealing with new business, they would adopt it to-morrow, if it would only necessitate the expenditure of such a comparatively small amount of time. Of course, the author advocated the system he brought forward, as one superseding the ordinary method of valuation, and in that respect it was superior to Mr. Searle's, because Mr. Searle did not put his method forward as a substitute for the ordinary method, but as a complement to it. The paper, which was a very voluminous one, embracing a very great amount of necessary details, was one that it was hardly possible to criticise or to form any final opinion about after the perusal of only a few days. The only test of the recommendations the author had placed before them was that of actual practice. He hoped the author, having brought before them his own views, would set the method into practice, and would, at a future day, tell the Institute

how it had worked. If he was able to tell them that in application practically to the needs of an insurance office the system did work out well, they would all be anxious to follow it out and adopt his suggestions.

On the motion of the President, a hearty vote of thanks was accorded to Mr. Chatham for his paper.

Mr. Chatham, in reply, thanked the members of the Institute for the kind manner in which they had received his essay. In reply to Mr. Higham's question, he might say his scheme *was* in actual use in one office, and others were adopting it. Mr. Higham had said that too much was expected from the books recommended, but he had not pointed out any particular in which they fell short of what was stated. He had also said there was too much detail in the essay, and afterwards referred to matters which he thought might be amplified, which was hardly consistent, as they were of very little importance. There was one thing in which he (Mr. Chatham) agreed with him, namely, the columns for lives which were not necessary, but were asked for in the syllabus. That was referred to in the essay. Mr. Higham had stated that the differentiation of the surrenders and lapses would break down in an office where there were no lapses. He (Mr. Chatham), did not agree with this, but it was a matter of small moment as there were very few offices where there were no lapses. With regard to Mr. Manly's remarks, he could assure him that in a number of offices the records were so well kept, that there would be no difficulty in applying the checks mentioned in the essay, and therefore no more labour would be necessary. Mr. Manly, had said that he would have liked the forms of such records to have been given, but it was very difficult if not impossible to give forms which would be applicable to every office. Mr. Manly had also said he was bold in his recommendations with regard to the means of arriving at the net premiums from the office premiums, but he thought his recommendations were not bolder than the circumstances warranted. There was, however, the alternative of inserting the net premiums. He was glad Mr. Chisholm thought that he would look favourably upon the scheme if it avoided the work of the valuation, and he thought the attitude he had assumed towards it the proper one, namely to test it and see the results.

ORIGINAL TABLES.

I. Annual and Single Endowment Assurance Premiums.

II. Values of Log D_{xx}, Log N_{xx}, and a_{xx}.

H^M $2\frac{1}{2}$ per-cent.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—The accompanying Tables of Single and Annual Premiums for Endowment Assurances were commenced some time ago, on the appearance of Messrs. King and Whittall's $2\frac{1}{2}$ per-cent Tables; but were laid aside. I have recently been afforded an opportunity of

[*See p.* 458.]

I.—*Annual and Single Premiums for Endowment Assurances of* 100, *payable in* n *years.*—H[M] $2\frac{1}{2}$ *per-cent.*

Age at Entry at side: "*n*" (the number of years' duration) at top and bottom of each column.

	4	5	6	7	8	9	10	11	
15	23·616	18·698	15·428	13·099	11·356	10·002	8·923	8·042	15
16	23·637	18·725	15·456	13·128	11·385	10·031	8·951	8·068	16
17	23·664	18·756	15·488	13·159	11·415	10·059	8·978	8·094	17
18	23·698	18·788	15·521	13·188	11·442	10·086	9·003	8·117	18
19	23·725	18·819	15·547	13·214	11·463	10·105	9·021	8·135	19
20	23·746	18·833	15·560	13·225	11·475	10·116	9·030	8·145	20
21	23·753	18·838	15·566	13·230	11·479	10·121	9·037	8·151	21
22	23·753	18·838	15·566	13·230	11·481	10·122	9·038	8·154	22
23	23·753	18·838	15·563	13·228	11·481	10·124	9·041	8·156	23
24	23·746	18·838	15·563	13·230	11·483	10·127	9·045	8·162	24
25	23·753	18·838	15·569	13·237	11·491	10·135	9·053	8·170	25
26	23·753	18·847	15·576	13·245	11·500	10·144	9·063	8·180	26
27	23·766	18·856	15·589	13·257	11·510	10·155	9·074	8·191	27
28	23·773	18·869	15·599	13·267	11·522	10·168	9·086	8·204	28
29	23·787	18·878	15·608	13·279	11·533	10·179	9·098	8·216	29
30	23·794	18·887	15·618	13·289	11·543	10·190	9·110	8·229	30
31	23·801	18·897	15·628	13·297	11·553	10·200	9·122	8·240	31
32	23·808	18·901	15·638	13·309	11·565	10·213	9·134	8·253	32
33	23·815	18·915	15·647	13·319	11·576	10·226	9·146	8·266	33
34	23·828	18·924	15·661	13·331	11·590	10·238	9·161	8·280	34
35	23·842	18·938	15·674	13·346	11·604	10·253	9·174	8·295	35
36	23·856	18·951	15·687	13·361	11·620	10·267	9·190	8·311	36
37	23·863	18·965	15·700	13·374	11·634	10·282	9·205	8·326	37
38	23·877	18·974	15·713	13·386	11·645	10·297	9·220	8·344	38
39	23·884	18·988	15·723	13·399	11·659	10·311	9·236	8·361	39
40	23·898	18·997	15·736	13·411	11·673	10·327	9·254	8·381	40
41	23·904	19·006	15·749	13·426	11·691	10·347	9·276	8·403	41
42	23·918	19·020	15·766	13·447	11·713	10·372	9·301	8·431	42
43	23·932	19·043	15·789	13·472	11·741	10·400	9·331	8·461	43
44	23·953	19·066	15·816	13·502	11·772	10·431	9·365	8·497	44
45	23·981	19·094	15·849	13·535	11·806	10·468	9·401	8·534	45
46	24·009	19·127	15·883	13·569	11·841	10·504	9·440	8·575	46
47	24·044	19·159	15·916	13·605	11·879	10·543	9·481	8·618	47
48	24·072	19·192	15·947	13·641	11·916	10·584	9·523	8·663	48
49	24·100	19·225	15·984	13·677	11·956	10·626	9·569	8·712	49
50	24·128	19·258	16·021	13·716	11·999	10·672	9·619	8·764	50
51	24·164	19·295	16·059	13·763	12·047	10·724	9·673	8·824	51
52	24·199	19·338	16·107	13·813	12·102	10·783	9·737	8·890	52
53	24·242	19·386	16·162	13·872	12·166	10·850	9·807	8·965	53
54	24·292	19·443	16·221	13·936	12·234	10·923	9·885	9·049	54
55	24·349	19·500	16·288	14·006	12·310	11·005	9·973	9·140	55
56	24·407	19·568	16·362	14·084	12·396	11·095	10·069	9·242	56
57	24·472	19·641	16·440	14·172	12·489	11·194	10·174	9·353	57
58	24·544	19·719	16·529	14·269	12·592	11·307	10·292	9·477	58
59	24·625	19·813	16·630	14·379	12·708	11·429	10·421	9·612	59
60	24·713	19·917	16·744	14·499	12·835	11·563	10·561	9·759	60
	4	5	6	7	8	9	10	11	

Annual and Single Premiums, &c.—(continued).

	12	13	14	15	16	17	18	19	
15	7·308	6·690	6·162	5·704	5·306	4·956	4·646	4·369	15
16	7·334	6·715	6·185	5·728	5·329	4·979	4·668	4·392	16
17	7·359	6·740	6·209	5·752	5·352	5·001	4·691	4·411	17
18	7·382	6·761	6·231	5·772	5·373	5·021	4·710	4·433	18
19	7·399	6·778	6·247	5·789	5·389	5·037	4·726	4·449	19
20	7·409	6·788	6·257	5·798	5·399	5·048	4·737	4·459	20
21	7·415	6·794	6·263	5·806	5·406	5·055	4·744	4·468	21
22	7·418	6·798	6·268	5·810	5·411	5·061	4·751	4·475	22
23	7·422	6·802	6·273	5·815	5·417	5·067	4·757	4·481	23
24	7·428	6·808	6·279	5·823	5·424	5·075	4·766	4·490	24
25	7·437	6·817	6·289	5·832	5·434	5·085	4·777	4·502	25
26	7·446	6·828	6·300	5·844	5·447	5·097	4·789	4·514	26
27	7·459	6·841	6·313	5·857	5·460	5·111	4·803	4·528	27
28	7·472	6·854	6·326	5·871	5·474	5·125	4·817	4·543	28
29	7·484	6·867	6·339	5·884	5·487	5·139	4·831	4·557	29
30	7·496	6·879	6·352	5·897	5·501	5·153	4·846	4·573	30
31	7·508	6·892	6·365	5·910	5·515	5·167	4·861	4·588	31
32	7·521	6·905	6·379	5·924	5·529	5·182	4·877	4·605	32
33	7·535	6·919	6·393	5·940	5·545	5·199	4·894	4·623	33
34	7·549	6·934	6·408	5·956	5·562	5·217	4·913	4·643	34
35	7·565	6·950	6·425	5·974	5·582	5·237	4·934	4·665	35
36	7·582	6·967	6·444	5·993	5·602	5·259	4·956	4·688	36
37	7·599	6·986	6·463	6·013	5·623	5·281	4·979	4·713	37
38	7·616	7·005	6·484	6·035	5·646	5·305	5·004	4·739	38
39	7·636	7·025	6·506	6·058	5·670	5·330	5·031	4·767	39
40	7·657	7·048	6·530	6·083	5·696	5·358	5·061	4·798	40
41	7·681	7·073	6·556	6·112	5·726	5·389	5·094	4·833	41
42	7·709	7·103	6·587	6·145	5·760	5·426	5·132	4·873	42
43	7·742	7·137	6·623	6·182	5·800	5·466	5·174	4·917	43
44	7·778	7·175	6·663	6·223	5·843	5·511	5·221	4·966	44
45	7·818	7·216	6·706	6·268	5·889	5·560	5·273	5·020	45
46	7·861	7·261	6·752	6·317	5·941	5·614	5·329	5·079	46
47	7·905	7·308	6·801	6·368	5·996	5·672	5·389	5·142	47
48	7·953	7·358	6·855	6·425	6·054	5·734	5·455	5·211	48
49	8·005	7·412	6·912	6·486	6·119	5·802	5·526	5·286	49
50	8·061	7·472	6·975	6·552	6·189	5·876	5·604	5·367	50
51	8·123	7·539	7·046	6·626	6·267	5·958	5·690	5·458	51
52	8·194	7·613	7·125	6·710	6·355	6·050	5·786	5·558	52
53	8·273	7·698	7·213	6·803	6·452	6·151	5·892	5·668	53
54	8·361	7·790	7·310	6·905	6·559	6·263	6·008	5·789	54
55	8·459	7·893	7·418	7·017	6·676	6·385	6·136	5·923	55
56	8·566	8·005	7·536	7·140	6·804	6·518	6·276	6·069	56
57	8·683	8·127	7·664	7·273	6·944	6·666	6·430	...	57
58	8·812	8·262	7·805	7·421	7·099	6·828	...	...	58
59	8·953	8·411	7·959	7·584	7·271	...	...	...	59
60	9·107	8·572	8·130	7·763	...	...	...	...	60
	12	13	14	15	16	17	18	19	

Annual and Single Premiums, &c.—(continued).

	20	21	22	23	24	25	26	27	
15	4·122	3·899	3·697	3·514	3·347	3·195	3·055	2·926	15
16	4·144	3·921	3·719	3·536	3·369	3·216	3·076	2·948	16
17	4·166	3·942	3·740	3·557	3·390	3·238	3·098	2·969	17
18	4·185	3·962	3·760	3·577	3·410	3·257	3·117	2·988	18
19	4·201	3·977	3·776	3·592	3·425	3·273	3·133	3·005	19
20	4·212	3·989	3·787	3·604	3·437	3·285	3·145	3·017	20
21	4·220	3·997	3·796	3·613	3·447	3·295	3·155	3·028	21
22	4·227	4·005	3·804	3·621	3·455	3·303	3·165	3·037	22
23	4·235	4·013	3·812	3·630	3·464	3·313	3·175	3·048	23
24	4·244	4·022	3·822	3·640	3·475	3·324	3·186	3·060	24
25	4·255	4·034	3·834	3·653	3·488	3·338	3·200	3·075	25
26	4·268	4·047	3·848	3·667	3·503	3·353	3·216	3·091	26
27	4·283	4·062	3·863	3·683	3·519	3·370	3·234	3·109	27
28	4·298	4·077	3·879	3·699	3·536	3·388	3·252	3·128	28
29	4·313	4·094	3·895	3·716	3·554	3·406	3·271	3·148	29
30	4·329	4·110	3·913	3·734	3·572	3·425	3·291	3·169	30
31	4·345	4·127	3·930	3·753	3·592	3·445	3·312	3·190	31
32	4·363	4·145	3·950	3·773	3·612	3·467	3·335	3·214	32
33	4·382	4·165	3·970	3·794	3·635	3·491	3·359	3·240	33
34	4·402	4·187	3·993	3·818	3·660	3·516	3·386	3·268	34
35	4·425	4·210	4·017	3·843	3·687	3·544	3·415	3·298	35
36	4·449	4·236	4·044	3·871	3·716	3·575	3·447	3·332	36
37	4·475	4·262	4·072	3·901	3·746	3·607	3·481	3·367	37
38	4·503	4·291	4·102	3·932	3·780	3·642	3·518	3·406	38
39	4·532	4·323	4·135	3·967	3·816	3·680	3·558	3·448	39
40	4·565	4·357	4·171	4·005	3·856	3·722	3·602	3·494	40
41	4·601	4·395	4·212	4·047	3·901	3·769	3·651	3·545	41
42	4·643	4·439	4·257	4·095	3·950	3·821	3·705	3·602	42
43	4·690	4·488	4·308	4·149	4·006	3·879	3·766	3·665	43
44	4·741	4·541	4·365	4·208	4·068	3·943	3·833	3·734	44
45	4·798	4·601	4·427	4·272	4·135	4·013	3·905	3·810	45
46	4·859	4·665	4·494	4·342	4·209	4·090	3·984	3·892	46
47	4·926	4·735	4·567	4·418	4·287	4·171	4·070	3·980	47
48	4·998	4·810	4·645	4·500	4·372	4·260	4·162	4·076	48
49	5·076	4·892	4·730	4·588	4·464	4·356	4·262	27	
							26		
50	5·161	4·981	4·823	4·686	4·566	4·462			
51	5·255	5·079	4·925	4·792	4·676	25		52	
52	5·359	5·188	5·039	4·911	24		53		
53	5·475	5·308	5·164	23		54		41·556	23
54	5·602	5·440	22		55		40·932	41·134	22
		21				40·322	40·517	40·734	21
55	5·741			56	39·707	39·895	40·107	40·341	20
	20		57						
		58		39·076	39·259	39·463	39·690	39·941	19
	59		38·407	38·585	38·785	39·005	39·246	39·512	18
		37·705	37·878	38·071	38·285	38·520	38·778	39·058	17
16	36·983	37·151	37·339	37·546	37·776	38·024	38·298	38·593	16
15	36·420	36·602	36·805	37·029	37·271	37·537	37·822	38·132	15
	59	58	57	56	55	54	53	52	

Annual and Single Premiums, &c.—(continued).

	28	29	30	31	32	33	34	35	
15	2·808	2·698	2·597	2·503	2·415	2·334	2·258	2·188	15
16	2·829	2·720	2·618	2·524	2·437	2·356	2·280	2·210	16
17	2·850	2·741	2·639	2·546	2·458	2·378	2·302	2·232	17
18	2·870	2·760	2·659	2·565	2·479	2·398	2·323	2·253	18
19	2·886	2·777	2·676	2·583	2·496	2·416	2·341	2·272	19
20	2·899	2·790	2·689	2·596	2·510	2·430	2·356	2·287	20
21	2·910	2·802	2·701	2·609	2·523	2·444	2·370	2·301	21
22	2·920	2·812	2·713	2·621	2·535	2·456	2·383	2·315	22
23	2·931	2·824	2·725	2·633	2·549	2·470	2·398	2·330	23
24	2·944	2·837	2·739	2·648	2·563	2·486	2·414	2·348	24
25	2·959	2·853	2·755	2·665	2·581	2·504	2·433	2·367	25
26	2·976	2·870	2·773	2·683	2·601	2·524	2·454	2·389	26
27	2·995	2·889	2·793	2·704	2·622	2·547	2·477	2·413	27
28	3·014	2·910	2·814	2·726	2·645	2·570	2·502	2·439	28
29	3·035	2·931	2·836	2·749	2·669	2·595	2·528	2·466	29
30	3·057	2·954	2·860	2·774	2·694	2·622	2·555	2·495	30
31	3·079	2·978	2·885	2·800	2·721	2·650	2·585	2·525	31
32	3·104	3·003	2·912	2·828	2·751	2·681	2·617	2·559	32
33	3·131	3·031	2·941	2·858	2·783	2·714	2·652	2·595	33
34	3·160	3·062	2·973	2·892	2·818	2·751	2·690	2·635	34
35	3·192	3·096	3·008	2·928	2·856	2·791	2·731	2·678	35
36	3·227	3·132	3·046	2·968	2·897	2·834	2·776	2·724	36
37	3·264	3·171	3·087	3·011	2·942	2·880	2·824	2·774	37
38	3·304	3·213	3·131	3·056	2·990	2·930	2·876	2·828	38
39	3·348	3·259	3·179	3·106	3·042	2·984	2·932	2·886	39
40	3·397	3·309	3·231	3·161	3·098	3·043	2·993	2·950	40
41	3·450	3·365	3·289	3·221	3·161	3·107	3·060	35	
42	3·509	3·426	3·353	3·287	3·230	3·179	34		
43	3·575	3·495	3·424	3·361	3·306	33		44	
44	3·647	3·569	3·501	3·441	32		45		
45	3·725	3·651	3·586	31				47·261	31
46	3·811	3·840	30			46	46·505	46·763	30
47	3·903	29		48	47	45·759	46·010	46·290	29
	28		49		45·022	45·266	45·537	45·837	28
		50		44·295	44·529	44·793	45·085	45·405	27
			43·578	43·805	44·061	44·341	44·654	44·993	26
	51	42·880	43·100	43·349	43·622	43·922	44·251	44·610	25
24	42·205	42·417	42·656	42·922	43·215	43·532	43·880	44·254	24
23	41·763	41·995	42·251	42·534	42·844	43·180	43·544	43·934	23
22	41·359	41·607	41·880	42·180	42·505	42·859	43·239	43·646	22
21	40·976	41·241	41·532	41·846	42·188	42·556	42·951	43·376	21
20	40·598	40·878	41·185	41·515	41·873	42·256	42·668	43·107	20
19	40·215	40·510	40·832	41·178	41·549	41·949	42·376	42·829	19
18	39·800	40·112	40·449	40·810	41·198	41·610	42·051	42·520	18
17	39·361	39·688	40·039	40·415	40·817	41·246	41·700	42·180	17
16	38·910	39·251	39·617	40·010	40·424	40·868	41·337	41·829	16
15	38·463	38·820	39·200	39·605	40·037	40·493	40·973	41·480	15
	51	50	49	48	47	46	45	44	

Annual and Single Premiums, &c.—(continued).

	36	37	38	39	40	41	42	43	
15	2·122	2·060	2·003	1·949	1·899	1·852	1·808	1·767	15
16	2·144	2·083	2·026	1·972	1·923	1·876	1·832	1·792	16
17	2·167	2·106	2·049	1·996	1·946	1·900	1·857	1·817	17
18	2·188	2·127	2·071	2·018	1·969	1·923	1·881	1·841	18
19	2·207	2·147	2·091	2·038	1·990	1·945	1·903	1·863	19
20	2·223	2·163	2·108	2·056	2·008	1·963	1·922	1·884	20
21	2·238	2·179	2·124	2·073	2·025	1·981	1·941	1·903	21
22	2·252	2·194	2·140	2·089	2·043	2·000	1·960	1·923	22
23	2·268	2·210	2·157	2·107	2·061	2·019	1·980	1·944	23
24	2·286	2·229	2·176	2·128	2·083	2·041	2·003	1·968	24
25	2·306	2·250	2·198	2·151	2·107	2·066	2·029	1·995	25
26	2·329	2·274	2·223	2·176	2·133	2·094	2·058	2·025	26
27	2·354	2·300	2·250	2·204	2·162	2·124	2·089	2·057	27
28	2·381	2·327	2·279	2·234	2·193	2·156	2·122	2·092	28
29	2·409	2·357	2·309	2·266	2·226	2·190	2·158	2·128	29
30	2·439	2·388	2·342	2·300	2·262	2·227	2·196	2·168	30
31	2·471	2·422	2·377	2·336	2·299	2·266	2·236	2·209	31
32	2·506	2·458	2·414	2·375	2·339	2·308	2·279	2·254	32
33	2·544	2·497	2·455	2·417	2·383	2·353	2·326	43	
34	2·585	2·540	2·499	2·463	2·431	2·402	42		
35	2·629	2·586	2·547	2·513	2·482	41			
36	2·678	2·636	2·599	2·566	40				
37	2·729	2·690	2·655	39					
38	2·785	2·748	38						
39	2·846	37							
	36								

							36		39
							37	53·849	39
						38	52·976	53·312	38
					39	52·117	52·444	52·810	37
				40	51·273	51·588	51·939	52·332	36
			41	50·439	50·744	51·085	51·461	51·878	35
		42	49·620	49·915	50·244	50·610	51·012	51·451	34
	43	48·817	49·102	49·422	49·776	50·163	50·588	51·051	33
32	48·032	48·307	48·615	48·959	49·334	49·746	50·193	50·676	32
31	47·527	47·827	48·159	48·522	48·919	49·354	49·822	50·327	31
30	47·054	47·373	47·727	48·112	48·532	48·985	49·473	50·000	30
29	46·600	46·941	47·315	47·720	48·158	48·634	49·144	49·690	29
28	46·168	46·529	46·922	47·349	47·807	48·300	48·829	49·395	28
27	45·754	46·137	46·549	46·993	47·471	47·983	48·529	49·112	27
26	45·361	45·761	46·193	46·656	47·151	47·683	48·246	48·846	26
25	44·995	45·415	45·863	46·344	46·858	47·405	47·985	48·602	25
24	44·659	45·095	45·561	46·059	46·590	47·154	47·749	48·378	24
23	44·356	44·815	45·293	45·805	46·351	46·929	47·539	48·183	23
22	44·085	44·551	45·051	45·578	46·139	46·732	47·354	48·010	22
21	43·829	44·312	44·824	45·366	45·941	46·544	47·180	47·846	21
20	43·576	44·071	44·598	45·154	45·739	46·356	47·002	47·680	20
19	43·310	43·822	44·361	44·929	45·527	46·154	46·812	47·500	19
18	43·015	43·539	44·090	44·671	45·280	45·920	46·588	47·288	18
17	42·690	43·227	43·790	44·383	45·002	45·654	46·334	47·044	17
16	42·351	42·900	43·476	44·080	44·712	45·373	46·063	46·785	16
15	42·015	42·573	43·161	43·776	44·419	45·090	45·793	46·522	15
	43	42	41	40	39	38	37	36	

Annual and Single Premiums, &c.—(continued).

	44	45	46	47	48	49	50	51	
15	1·729	1·693	1·660	1·628	1·599	1·573	1·548	1·525	15
16	1·751	1·719	1·686	1·655	1·627	1·600	1·576	1·553	16
17	1·779	1·745	1·712	1·682	1·654	1·629	1·605	1·583	17
18	1·804	1·770	1·738	1·709	1·682	1·657	1·634	1·613	18
19	1·827	1·794	1·762	1·734	1·707	1·683	1·661	1·641	19
20	1·848	1·815	1·785	1·757	1·731	1·708	1·686	1·667	20
21	1·868	1·836	1·807	1·780	1·755	1·732	1·712	1·693	21
22	1·889	1·858	1·829	1·803	1·779	1·758	1·738	1·720	22
23	1·911	1·881	1·854	1·828	1·805	1·784	1·766	1·749	23
24	1·936	1·907	1·880	1·856	1·834	1·814	1·797	1·781	24
25	1·964	1·936	1·910	1·887	1·866	1·847	1·831	51	
26	1·995	1·968	1·943	1·921	1·901	1·884	50		
27	2·028	2·002	1·979	1·958	1·939	49			
28	2·064	2·039	2·017	1·997	48				
29	2·102	2·079	2·058	47					
30	2·142	2·120	46						
31	2·156	45							
	44								

	35	34	33	32	31	30	29	28	
								61·539	47
							60·524	60·973	46
						59·517	59·949	60·432	45
					58·522	58·939	59·405	59·922	44
				57·544	57·946	58·398	58·895	59·444	43
			56·585	56·973	57·407	57·888	58·417	58·995	42
		55·649	56·024	56·444	56·907	57·417	57·976	58·583	41
40	54·739	55·100	55·505	55·954	56·446	56·985	57·571	58·205	40
39	54·198	54·588	55·022	55·498	56·017	56·583	57·195	57·856	39
38	53·690	54·110	54·568	55·071	55·617	56·210	56·849	57·534	38
37	53·212	53·656	54·144	54·671	55·244	55·861	56·522	57·224	37
36	52·761	53·229	53·741	54·293	54·890	55·532	56·217	56·951	36
35	52·332	52·824	53·361	53·937	54·556	55·222	55·932	56·685	35
34	51·929	52·446	53·002	53·602	54·246	54·932	55·663	56·439	34
33	51·551	52·090	52·671	53·293	53·956	54·663	55·415	56·210	33
32	51·200	51·761	52·363	53·005	53·690	54·417	55·185	55·998	32
31	50·871	51·454	52·076	52·737	53·441	54·185	54·973	55·802	31
30	50·563	51·166	51·807	52·488	53·210	53·971	54·773	55·620	30
29	50·273	50·893	51·554	52·251	52·988	53·766	54·583	55·441	29
28	49·995	50·634	51·310	52·024	52·778	53·571	54·402	55·276	28
27	49·732	50·388	51·080	51·810	52·576	53·383	54·227	55·112	27
26	49·483	50·154	50·859	51·602	52·385	53·202	54·061	54·959	26
25	49·251	49·937	50·659	51·415	52·210	53·039	53·910	54·817	25
24	49·044	49·741	50·476	51·244	52·051	52·893	53·771	54·690	24
23	48·861	49·571	50·317	51·098	51·912	52·766	53·654	54·580	23
22	48·700	49·422	50·178	50·968	51·795	52·656	53·554	54·490	22
21	48·546	49·281	50·046	50·846	51·683	52·551	53·459	54·402	21
20	48·393	49·134	49·910	50·720	51·563	52·441	53·350	54·307	20
19	48·222	48·973	49·759	50·578	51·429	52·317	53·239	54·200	19
18	48·017	48·780	49·576	50·402	51·263	52·161	53·090	54·059	18
17	47·785	48·556	49·361	50·198	51·068	51·973	52·915	53·888	17
16	47·534	48·317	49·132	49·978	50·859	51·773	52·720	53·702	16
15	47·283	48·076	48·900	49·756	50·644	51·566	52·524	53·515	15
	35	34	33	32	31	30	29	28	

Annual and Single Premiums, &c.—(continued).

	52	53	54	55	56	57	58	59	
15	1·503	1·484	1·466	1·449	1·434	1·420	1·408	1·397	15
16	1·533	1·514	1·496	1·481	1·466	1·453	1·441	1·431	16
17	1·563	1·545	1·528	1·513	1·499	1·487	1·476	59	
18	1·593	1·576	1·560	1·545	1·532	1·521	58	20	
19	1·622	1·605	1·590	1·576	1·564	57			
20	1·649	1·633	1·619	1·606	56		21		
21	1·676	1·661	1·648	55		22			
22	1·704	1·690	54		23				
23	1·734	53		24					
	52		25						
		26							
	27								
								70·183	55
							69·044	69·666	54
						67·922	68·517	69·180	53
					66·815	67·383	68·020	68·724	52
				65·724	66·271	66·880	67·556	68·300	51
			64·656	65·180	65·766	66·415	67·129	67·910	50
		63·602	64·107	64·671	65·293	65·978	66·729	67·544	49
48	62·566	63·049	63·590	64·190	64·849	65·571	66·354	67·202	48
47	62·005	62·527	63·102	63·737	64·432	65·185	66·002	66·883	47
46	61·473	62·029	62·641	63·310	64·034	64·822	65·668	66·580	46
45	60·968	61·556	62·200	62·900	63·659	64·476	65·354	66·298	45
44	60·490	61·110	61·785	62·515	63·305	64·151	65·059	66·029	44
43	60·041	60·693	61·398	62·159	62·976	63·851	64·788	65·785	43
42	59·624	60·305	61·039	61·827	62·673	63·576	64·539	65·559	42
41	59·239	59·949	60·710	61·527	62·398	63·327	64·312	65·356	41
40	58·888	59·624	60·412	61·254	62·151	63·102	64·112	65·176	40
39	58·568	59·329	60·141	61·007	61·927	62·900	63·929	65·012	39
38	58·271	59·056	59·893	60·780	61·720	62·715	63·761	64·863	38
37	57·993	58·802	59·659	60·568	61·529	62·541	63·605	64·722	37
36	57·734	58·563	59·441	60·371	61·349	62·378	63·459	64·590	36
35	57·488	58·339	59·237	60·183	61·178	62·222	63·320	64·466	35
34	57·261	58·129	59·044	60·007	61·020	62·078	63·188	64·349	34
33	57·051	57·937	58·868	59·846	60·871	61·944	63·068	64·241	33
32	56·856	57·756	58·702	59·695	60·734	61·822	62·956	64·141	32
31	56·673	57·590	58·551	59·556	60·610	61·707	62·854	64·049	31
30	56·505	57·434	58·410	59·427	60·490	61·600	62·756	63·961	30
29	56·344	57·285	58·273	59·302	60·376	61·495	62·663	63·876	29
28	56·188	57·144	58·141	59·180	60·266	61·395	62·571	63·795	28
27	56·039	57·005	58·012	59·063	60·159	61·298	62·483	63·715	27
26	55·895	56·871	57·890	58·951	60·054	61·202	62·398	63·637	26
25	55·763	56·751	57·778	58·849	59·961	61·117	62·320	63·566	25
24	55·646	56·641	57·680	58·759	59·878	61·044	62·251	63·502	24
23	55·546	56·551	57·598	58·683	59·812	60·980	62·195	63·454	23
22	55·463	56·476	57·527	58·622	59·754	60·932	62·149	63·412	22
21	55·383	56·402	57·463	58·561	59·700	60·883	62·105	63·373	21
20	55·298	56·324	57·388	58·493	59·639	60·824	62·054	63·327	20
19	55·195	56·229	57·300	58·410	59·561	60·754	61·988	63·266	19
18	55·061	56·102	57·180	58·298	59·456	60·654	61·895	63·180	18
17	54·900	55·949	57·034	58·159	59·324	60·529	61·778	63·071	17
16	54·722	55·778	56·873	58·005	59·178	60·393	61·649	62·949	16
15	54·541	55·607	56·707	57·849	59·032	60·254	61·517	62·824	15
	27	26	25	24	23	22	21	20	

Annual and Single Premiums, &c.—(continued).

	19	18	17	16	15	14	13	12	
								79·834	63
							78·568	79·495	62
						77·322	78·198	79·176	61
					76·093	76·922	77·849	78·876	60
				74·880	75·666	76·544	77·520	78·590	59
			73·680	74·429	75·263	76·190	77·207	78·322	58
		72·500	73·212	74·007	74·888	75·859	76·917	78·071	57
56	71·334	72·012	72·771	73·612	74·537	75·549	76·646	77·837	56
55	70·832	71·556	72·359	73·241	74·207	75·256	76·393	77·620	55
54	70·359	71·127	71·971	72·893	73·898	74·983	76·156	77·417	54
53	69·915	70·724	71·607	72·568	73·610	74·732	75·939	77·232	53
52	69·500	70·346	71·268	72·266	73·341	74·498	75·737	77·061	52
51	69·115	69·998	70·954	71·985	73·095	74·285	75·556	76·907	51
50	68·756	69·676	70·666	71·732	72·873	74·093	75·390	76·771	50
49	68·427	69·378	70·402	71·500	72·671	73·917	75·241	76·646	49
48	68·117	69·102	70·156	71·283	72·483	73·756	75·105	76·529	48
47	67·829	68·844	69·929	71·083	72·307	73·605	74·976	76·422	47
46	67·559	68·602	69·715	70·895	72·144	73·463	74·856	76·320	46
45	67·302	68·373	69·510	70·715	71·988	73·329	74·739	76·222	45
44	67·063	68·159	69·322	70·549	71·841	73·202	74·629	76·127	44
43	66·841	67·963	69·146	70·395	71·707	73·085	74·529	76·044	43
42	66·641	67·783	68·988	70·254	71·585	72·978	74·439	75·966	42
41	66·459	67·622	68·844	70·129	71·476	72·885	74·359	75·900	41
40	66·298	67·478	68·720	70·020	71·380	72·805	74·290	75·841	40
39	66·154	67·351	68·607	69·922	71·295	72·732	74·229	75·790	39
38	66·020	67·232	68·502	69·832	71·217	72·666	74·173	75·744	38
37	65·895	67·122	68·405	69·746	71·144	72·602	74·122	75·702	37
36	65·778	67·020	68·315	69·666	71·076	72·544	74·071	75·661	36
35	65·666	66·920	68·227	69·590	71·007	72·485	74·022	75·620	35
34	65·561	66·827	68·144	69·517	70·946	72·432	73·978	75·580	34
33	65·463	66·739	68·068	69·451	70·890	72·383	73·937	75·546	33
32	65·376	66·661	67·998	69·390	70·837	72·339	73·898	75·512	32
31	65·293	66·588	67·934	69·334	70·788	72·295	73·861	75·480	31
30	65·215	66·520	67·873	69·280	70·741	72·256	73·824	75·451	30
29	65·139	66·451	67·815	69·229	70·695	72·215	73·790	75·420	29
28	65·066	66·385	67·756	69·176	70·649	72·173	73·754	75·390	28
27	64·993	66·320	67·695	69·122	70·600	72·132	73·717	75·359	27
26	64·922	66·256	67·637	69·071	70·554	72·090	73·680	75·327	26
25	64·859	66·198	67·585	69·022	70·512	72·054	73·649	75·302	25
24	64·802	66·146	67·539	68·983	70·478	72·024	73·624	75·280	24
23	64·756	66·107	67·505	68·954	70·451	72·002	73·607	75·266	23
22	64·722	66·076	67·478	68·932	70·434	71·988	73·595	75·256	22
21	64·688	66·046	67·454	68·910	70·417	71·973	73·583	75·249	21
20	64·644	66·010	67·422	68·880	70·390	71·951	73·566	75·234	20
19	64·590	65·959	67·376	68·841	70·356	71·920	73·537	75·207	19
18	64·510	65·885	67·307	68·778	70·298	71·868	73·490	75·166	18
17	64·407	65·790	67·217	68·695	70·222	71·798	73·427	75·107	17
16	64·293	65·683	67·120	68·602	70·137	71·720	73·356	75·044	16
15	64·176	65·573	67·017	68·510	70·049	71·641	73·283	74·978	15
	19	18	17	16	15	14	13	12	

Annual and Single Premiums, &c.—(continued).

	11	10	9	8	7	6	5	4	
64	81·122	82·171	83·344	84·641	86·068	87·629	89·327	91·163	64
63	80·820	81·920	83·139	84·480	85·944	87·539	89·263	91·124	63
62	80·532	81·680	82·944	84·324	85·824	87·449	89·202	91·088	62
61	80·261	81·454	82·756	84·176	85·710	87·363	89·144	91·051	61
60	80·005	81·239	82·580	84·032	85·600	87·285	89·090	91·017	60
59	79·761	81·034	82·412	83·898	85·498	87·210	89·039	90·988	59
58	79·532	80·841	82·256	83·773	85·402	87·141	88·993	90·961	58
57	79·317	80·663	82·110	83·661	85·317	87·080	88·954	90·937	57
56	79·120	80·500	81·978	83·559	85·239	87·027	88·917	90·915	56
55	78·937	80·349	81·859	83·463	85·168	86·976	88·883	90·895	55
54	78·768	80·210	81·746	83·378	85·105	86·929	88·854	90·876	54
53	78·612	80·083	81·646	83·300	85·046	86·888	88·824	90·859	53
52	78·471	79·968	81·554	83·227	84·993	86·849	88·800	90·844	52
51	78·344	79·863	81·471	83·163	84·946	86·815	88·778	90·832	51
50	78·229	79·773	81·398	83·107	84·902	86·788	88·759	90·819	50
49	78·127	79·688	81·332	83·056	84·866	86·761	88·741	90·810	49
48	78·032	79·610	81·271	83·010	84·832	86·734	88·724	90·800	48
47	77·941	79·539	81·212	82·966	84·798	86·712	88·707	90·790	47
46	77·856	79·468	81·156	82·920	84·763	86·688	88·690	90·778	46
45	77·773	79·400	81·102	82·878	84·732	86·663	88·673	90·768	45
44	77·698	79·337	81·049	82·837	84·700	86·639	88·659	90·759	44
43	77·624	79·278	81·002	82·800	84·671	86·619	88·646	90·751	43
42	77·561	79·224	80·961	82·766	84·646	86·602	88·634	90·746	42
41	77·505	79·180	80·924	82·739	84·627	86·590	88·627	90·741	41
40	77·459	79·141	80·895	82·717	84·612	86·580	88·622	90·739	40
39	77·417	79·110	80·871	82·700	84·600	86·571	88·617	90·734	39
38	77·380	79·080	80·849	82·683	84·588	86·563	88·610	90·732	38
37	77·344	79·054	80·827	82·668	84·576	86·554	88·605	90·727	37
36	77·312	79·027	80·805	82·651	84·563	86·544	88·598	90·724	36
35	77·278	78·998	80·783	82·632	84·549	86·534	88·590	90·720	35
34	77·246	78·973	80·761	82·615	84·534	86·524	88·583	90·715	34
33	77·217	78·946	80·741	82·598	84·522	86·515	88·578	90·710	33
32	77·188	78·924	80·722	82·583	84·512	86·507	88·571	90·707	32
31	77·161	78·902	80·702	82·568	84·500	86·500	88·568	90·705	31
30	77·137	78·880	80·688	82·556	84·493	86·493	88·563	90·702	30
29	77·110	78·859	80·671	82·544	84·483	86·485	88·559	90·700	29
28	77·083	78·837	80·654	82·529	84·471	86·478	88·554	90·695	28
27	77·056	78·815	80·634	82·515	84·461	86·471	88·546	90·693	27
26	77·032	78·795	80·617	82·502	84·449	86·461	88·541	90·688	26
25	77·010	78·776	80·602	82·490	84·441	86·456	88·537	90·688	25
24	76·993	78·761	80·590	82·480	84·434	86·451	88·537	90·685	24
23	76·980	78·754	80·585	82·478	84·432	86·451	88·537	90·688	23
22	76·976	78·749	80·583	82·478	84·434	86·454	88·537	90·688	22
21	76·968	78·746	80·580	82·476	84·434	86·454	88·537	90·688	21
20	76·956	78·734	80·573	82·471	84·429	86·448	88·534	90·685	20
19	76·934	78·717	80·556	82·456	84·417	86·439	88·527	90·678	19
18	76·895	78·683	80·527	82·429	84·393	86·420	88·510	90·668	18
17	76·844	78·637	80·485	82·395	84·363	86·395	88·493	90·656	17
16	76·788	78·585	80·441	82·356	84·332	86·371	88·476	90·646	16
15	76·729	78·534	80·395	82·320	84·302	86·349	88·461	90·639	15
	11	10	9	8	7	6	5	4	

II.—*Values of* Log D_{xx}, Log N_{xx}, *and* a_{xx}, *for Two Lives of Equal Age*—H^M $2\frac{1}{2}$ *per-cent.*

x	Log D_{xx}	Log N_{xx}	a_{xx}	x	Log D_{xx}	Log N_{xx}	a_{xx}
10	5·892761	7·253460	22·946	55	5·055999	6·019805	9·200
11	·877771	·234778	22·751	56	·026810	5·973262	8·840
12	·863575	·215899	22·507	57	4·996362	·924860	8·482
13	·849963	·196784	22·224	58	·964546	·874461	8·127
14	·836703	·177401	21·913	59	·931274	·821907	7·774
15	·823577	·157736	21·585	60	·896296	·767039	7·426
16	·810356	·137768	21·253	61	·859404	·709692	7·084
17	·796808	·117495	20·926	62	·820396	·649694	6·750
18	·782704	·096924	20·617	63	·779054	·586862	6·424
19	·767808	·076077	20·336	64	·735140	·521009	6·108
20	·752080	·054976	20·086	65	·688586	·451914	5·799
21	·735840	·033632	19·851	66	·639297	·379309	5·496
22	·719256	·012047	19·624	67	·587151	·302876	5·197
23	·702566	6·990211	19·393	68	·531977	·222229	4·901
24	·685948	·968104	19·149	69	·473737	·136865	4·604
25	·669438	·945703	18·891	70	·411723	·046260	4·311
26	·652937	·922987	18·623	71	·345227	4·949864	4·024
27	·636387	·899938	18·346	72	·273289	·847170	3·749
28	·619645	·876542	18·067	73	·194905	·737754	3·490
29	·602669	·852788	17·788	74	·109051	·621345	3·253
30	·585465	·828664	17·507	75	·015270	·497735	3·037
31	·568007	·804154	17·225	76	3·914612	·366296	2·829
32	·550377	·779241	16·938	77	·806200	·226416	2·632
33	·532585	·753905	16·646	78	·689672	·077324	2·411
34	·514635	·728122	16·349	79	·564736	3·917936	2·255
35	·496493	·701867	16·046	80	·429990	·747158	2·076
36	·478116	·675116	15·740	81	·283552	·564149	1·908
37	·459444	·647846	15·431	82	·123416	·368652	1·759
38	·440460	·620034	15·121	83	2·949434	·160431	1·626
39	·421200	·591651	14·806	84	·760112	2·940161	1·514
40	·401674	·562663	14·487	85	·556788	·708322	1·418
41	·381952	·533028	14·160	86	·341443	·464459	1·327
42	·362072	·502696	13·824	87	·115293	·206758	1·234
43	·341974	·471613	13·478	88	1·876163	1·933407	1·141
44	·321532	·439731	13·128	89	·627867	·636842	1·021
45	·300710	·406995	12·773	90	·363555	·306172	·876
46	·279333	·373357	12·417	91	·068157	0·931407	·730
47	·257295	·338771	12·064	92	0·731677	·498021	·584
48	·234587	·303185	11·711	93	·345023	$\bar{1}$·970668	·422
49	·211229	·266540	11·358	94	$\bar{1}$·867455	·296033	·268
50	·187185	·228767	11·005	95	·241897	$\bar{2}$·364964	·133
51	·162495	·189786	10·649	96	$\bar{2}$·350898	$\bar{1}$·868268	·033
52	·137169	·149500	10·288	97	$\bar{4}$·868268		
53	·111067	·107810	9·925				
54	·084035	·064616	9·563				

completing the tables in the course of some official investigations, and now offer them as a contribution to the *Journal*, in the hope that they may prove of some use.

The premiums have been calculated by the ordinary formulas, $A=1-d(1+a)$ and $P=\dfrac{1}{1+a}-d$, and checked by entering Messrs. Rothery and Ryan's Conversion Tables with the annuity-values.

The form of tabulation is the same as that used by Mr. Ambrose Smith in the similar 3 per-cent Tables (*J.I.A.*, xxii, 143).

The Table of Joint-Life Annuities and Commutation Columns is added, as it will serve for comparison, though it may probably not be of much general use.

I am, Sir,
Your obedient Servant,
J. McDONALD.

Prudential Assurance Company,
June 1896.

CORRESPONDENCE.

ON THE FORMULÆ FOR COMPLETE ANNUITIES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—I have often been struck by the difficulty of the proofs given in Part II, Chap. XI, of the *Text-Book* for the correction to be made to complete the value of an annuity. This difficulty seems to me to arise from our confining ourselves too much to the idea of a year as the unit of time, forgetting that, so far as life assurance is concerned, a year is an arbitrary interval taken for the sake of convenience in computation.

There are two principal formulæ proved in Chap. XI, from which all the others given may be at once derived. The first, on the assumption of a uniform distribution of deaths, is proved in sec. (5), and is:

$$\mathring{a}_x = a_x + \bar{A}_x \frac{i-\delta}{i\delta}.$$

In this equation change the unit of time from one year to $\frac{1}{m}$ of a year, then $\mathring{a}_x$ becomes $m\mathring{a}_x^{(m)}$, because 1 is payable at the end of each interval, a_x becomes $ma_x^{(m)}$, $\bar{A}_x$ is unchanged, i becomes $\frac{j_{(m)}}{m}$, and δ, which is equal to $\log(1+i)$, becomes $\log\left(1+\frac{j_{(m)}}{m}\right)=\frac{1}{m}\log(1+i)=\frac{\delta}{m}$, and the equation becomes

$$m\mathring{a}_x^{(m)} = ma_x^{(m)} + \bar{A}_x \frac{\dfrac{j_{(m)}}{m}-\dfrac{\delta}{m}}{\dfrac{j_m\delta}{m^2}}$$

or
$$\bar{a}_x^{(m)} = a_x^{(m)} + \bar{A}_x \frac{j_{(m)} - \delta}{j_m \delta},$$

which is equation (7) in section (8).

The second formula, which is independent of any assumption as to the distribution of deaths, is proved in sec. (11), and is:

$$\bar{a}_x = a_x + \frac{\bar{M}_x}{2D_x} - \frac{\bar{M}_x - \bar{M}_{x+1}}{12D_x}.$$

Changing the unit of time in this equation, we get

$$m\bar{a}_x^{(m)} = ma_x^{(m)} + \frac{\bar{M}_x}{2D_x} - \frac{\bar{M}_x - \bar{M}_{x+\frac{1}{m}}}{12D_x}.$$

Take as a first approximation to the value of $\bar{M}_{x+\frac{1}{m}}$,

$$\bar{M}_x + \frac{1}{m}(\bar{M}_{x+1} - \bar{M}_x);$$

then, dividing the above equation by m, we get at once

$$\bar{a}_x^{(m)} = a_x^{(m)} + \frac{1}{2m}\bar{A}_x - \frac{1}{12m^2}A_{x1}^{1},$$

which is equation (9) of section (12).

A more accurate approximation to the value of $\bar{M}_{x+\frac{1}{m}}$, however, is

$$\bar{M}_x + \frac{1}{m}\frac{d\bar{M}_x}{dx} = \bar{M}_x - \frac{\mu_x D_x}{m}.$$

$$\therefore \quad \frac{\bar{M}_x - \bar{M}_{x+\frac{1}{m}}}{12D_x} = \frac{\mu_x}{12m},$$

and we get
$$\bar{a}_x^{(m)} = a_x^{(m)} + \frac{1}{2m}\bar{A}_x - \frac{\mu_x}{12m^2},$$

which is equation (11) of section (18).

In this way the connection between the various formulæ is shown. I may mention, in passing, that in the proof of the second principal equation, on page 189, the value of the correction for the first tth part of the year is only $\frac{1}{t}\left(\bar{M} - \bar{M}_{\frac{1}{t}}\right)$ in the limit when t is infinite, and there should be a foot-note to that effect, as there is on page 186.

I am, Sir,

Your obedient servant,

H. N. SHEPPARD.

3 *Chichester Road, London, W.*,
10 *March* 1896.

ON A LIFE INTEREST IN AN ESTATE INCLUDING A REVERSION EXPECTANT ON THE DEATH OF THE LIFE TENANT IN THE ESTATE.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—It not infrequently happens that an estate, left by will, to be enjoyed by one person for life and by another afterwards, includes a reversionary interest. Sometimes, as in the case communicated by Dr. Sprague to the *Journal* in Jan. 1892 (*J.I.A.*, xxix, 540), the person who has the life interest in the estate of the testator is the same life as that upon which the reversion depends, but more generally these persons are different. A recent decision of the court with regard to cases of this sort, which has been brought under my notice, may, perhaps, be of interest to some readers of the *Journal*. The rule of law which is generally applicable to these cases is that laid down in Howe *v*. Lord Dartmouth, 7 Ves. 137, according to which, if the reversion had not been converted at the testator's death, the reversioner gets the sum which, at 4 per-cent compound interest (with yearly rests and deducting income tax), will accumulate from such date to the sum realized by sale or falling in of the reversion, and the life tenant (in the reversion) gets the balance; but it has been decided that, where it can be gathered from the will that the testator intended that conversion is to take place at some time other than that at which the rule of the court would make conversion necessary at the testator's death, the rule of the court as to division has no application, and the person who has been left a life interest in the estate is not entitled to any part of the proceeds of the reversion, if it is realized or sold after the death of such person. The particular case which came under my notice was *In re* Pitcairn, Brandreth *v*. Colvin, 44 W.R. 200, in which the testator, by his will, gave his trustees power, if and when they considered it expedient, to sell all and any part of his estate; and it was decided that he had shown an intention that a conversion of his estate (which included a reversionary interest) should take place at some time, which need not be that at which the rule of the court applied. The trustees had not converted the reversion, but on the death of the tenant for life in the estate, the reversion fell in (this tenant for life in the estate of the testator being the same life as that on which the reversion depended), and her executors received nothing in respect of income in the proceeds. The reason for this decision as regards reversions seems to be that, as in the case of leaseholds, terminable annuities, or other perishable property, it has been decided that, where the testator has shown an intention that the rule of the court is not to apply, the tenant for life is to enjoy them *in specie;* so, on the other hand, reversions, or other deferred property are to be similarly treated, and the tenant for life may get nothing from them.

I am,
Your obedient servant,
J. R. HART.

69 *King William Street*,
9 *June* 1896.

THE LIFE ASSURANCE COMPANIES OF THE UNITED KINGDOM.

Summary of the Life Assurance and Annuity Revenue Accounts.

[Extracted from the Parliamentary Return for 1895, published in 1896.]

INCOME	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Balance at the beginning of the Year	188,372,536	11,875,423	200,247,959
Premiums	17,637,683	6,382,927	24,020,610
Consideration for Annuities	1,742,387	10,550	1,752,937
Interest and Dividends (less Tax)	7,393,739	396,850	7,790,589
Increase in value of Investments	198,286	953	199,239
Fines, Fees, &c.	11,106	996	12,102
Capital Paid-up	368	101,488	101,856
Customs Timber Measuring, &c.	3,122	...	3,122
Donations (Itinerant Methodists)	356	...	356
Transfers from other Accounts	28,160	41,590	69,750
Miscellaneous	784	...	784
	215,388,527	18,813,777	234,202,304

OUTGO	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Claims	12,792,252	2,418,754	15,211,006
Cash Bonuses and Reduction of Premiums	1,054,089	13	1,054,102
Surrenders	981,465	25,220	1,006,685
Annuities	1,135,282	3,806	1,139,088
Commission	990,651	1,708,887	2,699,538
Expenses of Management	1,593,984	1,080,028	2,674,012
Bad Debts	2,486	208	2,694
Decrease in value of Investments	146,349	...	146,349
Interest on Capital and Dividends and Bonuses to Shareholders	616,696	286,119	902,815
Transfers to other Accounts	33,242	673	33,915
Miscellaneous	31,648	17	31,665
Balance* at the end of the Year	196,010,383	13,290,052	209,300,435
	215,388,527	18,813,777	234,202,304

* This Balance includes the whole of the Life and Annuity Funds (£203,391,610), and, in addition, the Capital of Companies whose business is limited to Life Assurance only.

Summary of the Balance Sheets (1895).

Liabilities	Ordinary Companies	Industrial Companies	Total
	£	£	£
Paid-up Capital (including sundry Shareholders' Balances)	11,120,920	1,106,085	12,227,005
Life and Annuity Funds	190,918,237	12,473,373	203,391,610
Fire Funds of Companies transacting Life Business	9,753,995	...	9,753,995
Marine Funds of Companies transacting Life Business	635,427	...	635,427
Reserve Funds	4,070,823	...	4,070,823
Other Funds	756,258	196,323	952,581
Profit and Loss Balances	3,280,762	...	3,280,762
Depreciation and Investment Balances	892,919	877	893,796
Globe Annuitants (Liverpool and London)	1,102,800	...	1,102,800
Outstanding Claims	3,445,349	11,427	3,456,776
Outstanding Accounts	459,053	15,142	474,195
Temporary Loans	158,096	...	158,096
	226,594,639	13,803,227	240,397,866

Assets	Ordinary Companies	Industrial Companies	Total
	£	£	£
Mortgages	84,370,679	466,752	84,837,431
Loans on Policies	10,615,387	33,848	10,649,235
,, Rates	22,527,487	5,417,995	27,945,482
British Government Securities	4,922,976	793,950	5,716,926
Indian and Colonial Government Securities	16,886,608	225,926	17,112,534
Foreign Government Securities	4,602,789	...	4,602,789
Debentures	29,010,542	1,766,868	30,777,410
Shares and Stocks	16,226,370	28,053	16,254,423
Companies' own Shares	620,709	...	620,709
Land and House Property and Ground Rents	15,490,108	3,793,261	19,283,369
Life Interests and Reversions	4,805,687	433	4,806,120
Loans on Personal Security	1,484,530	7,283	1,491,813
Agents' Balances and Outstanding Premiums	4,935,097	491,270	5,426,367
Outstanding Interest	2,180,990	137,597	2,318,587
Cash, Deposits, Stamps, &c.	7,741,602	245,860	7,987,462
Customs Timber Measuring Balances, &c.	3,442	...	3,442
Book-Room Grant (Itinerant Methodists)	25,000	...	25,000
Deficiencies, Preliminary Expenses, &c.	144,636	394,131	538,767
	226,594,639	13,803,227	240,397,866

INCREASE (+) *or* DECREASE (−) *in the Chief Items of this Year's* SUMMARY *compared with the corresponding Items for the previous Year*

	Ordinary Companies	Industrial Companies
	£	£
INCOME.		
Premiums	+ 775,169	− 455,092
Consideration for Annuities	+ 326,618	− 9,759
Interest and Dividends (less Tax)	− 140,992	− 28,211
Net Result of Realization and Re-valuation of Investments	− 102,557	− 2,577
OUTGO.		
Claims	− 17,323	− 129,078
Annuities	+ 81,006	− 1,388
Surrenders	− 25,386	− 1,827
Commission	− 74,345	+ 131,084
Expenses of Management	− 5,861	− 85,101
LIABILITIES.		
Paid-up Capital (including sundry Shareholders' Balances)	− 77,026	− 142,582
Life and Annuity Funds	+ 7,908,533	+ 1,307,648
ASSETS.		
Mortgages (including Loans on Rates)	+ 734,933	+ 701,878
Life Interests and Reversions	+ 654,813	− 6
Loans on Policies	− 344,878	+ 4,631
British Government Securities	− 54,904	− 159,891
Indian and Colonial Government Securities	+ 1,232,137	− 62,341
Foreign Government Securities	− 440,450	...
Debentures	+ 1,997,555	+ 9,830
Shares and Stocks	+ 1,810,703	− 8,841
Companies' own Shares	− 14,294	...
Land and House Property and Ground Rents	+ 708,621	+ 265,422
Loans on Personal Security	− 66,724	− 994

NUMBER OF COMPANIES.

The total number of Companies appearing in the above Summary is 92, of which 81 have been classed as Ordinary, 7 as Industrial, and 4 appear in both Classes, the Returns of these Companies showing the Ordinary and Industrial business separately.

During the year three names have been removed from the official List of Companies, namely, London Amicable, Limited; West of England; Whittington; their business having been transferred.

SUMMARY OF THE ASSURANCES IN FORCE, *as shown by the last Returns of the Compani*

ORDINARY BUSINESS.

	With Profits		Without Profits		Total		Re-assurances	Net Amount
	No.	Amount	No.	Amount	No.	Amount	Amount	
Assurances.		£		£		£	£	£
Whole Term of Life	739,059	360,068,742	124,324	62,472,299	863,383	422,541,041	21,412,078	401,128,9
Limited number of Premiums . .	33,472	20,269,916	5,978	2,777,475	39,450	23,047,391	832,217	22,215,1
	772,531	380,338,658	130,302	65,249,774	902,833	445,588,432	22,244,295	423,344,1
Endowments . .	1,929	419,872	13,146	2,602,133	15,075	3,022,005	3,000	3,019,0
Endowment Assurances . . .	149,337	76,208,728	32,435	10,134,612	181,772	86,343,340	1,198,675	85,144,6
Joint Lives . .	14,119	2,845,707	2,252	1,057,703	16,371	3,903,410	361,636	3,541,7
Last Survivor . .	1,075	874,193	1,223	1,379,919	2,298	2,254,112	353,549	1,900,5
Contingent . .	25	24,536	3,408	5,152,329	3,433	5,176,865	1,096,071	4,080,7
Issue . . .	2	3,312	958	3,914,084	960	3,917,396	1,144,803	2,772,5
Miscellaneous . .	507	391,774	4,888	6,510,680	5,395	6,902,454	1,521,641	5,380,8
	1,239,525	461,106,780	188,612	96,001,234	1,428,137	557,108,014	27,923,670	529,184,3
Annuities.								
Immediate . .	...	...	...	...	21,342	1,027,233	11,260	1,015,9
Deferred . .	...	...	...	...	6,688	196,505	10,880	185,6
	...	...	...	...	28,030	1,223,738	22,140	1,201,5

INDUSTRIAL BUSINESS—(Sickness and Friendly Society Contracts not included).

	With Profits		Without Profits		Total		Re-assurances	Net Amount
	No.	Amount	No.	Amount	No.	Amount	Amount	
Assurances.						£		£
Whole Term of Life	...	...	...	...	14,475,240	137,086,694	...	137,086,6
Limited number of Premiums . .	...	...	...	...	453	3,528	...	3,5
	...	...	...	...	14,475,693	137,090,222	...	137,090,2
Endowments . .	...	...	...	...	103,283	1,433,620	...	1,433,6
Endowment Assurances . . .	...	...	...	...	157,849	1,612,181	...	1,612,1
Joint Lives . .	...	...	...	...	253,756	4,006,546	...	4,006,5
	...	...	...	...	14,990,581	144,142,569	...	144,142,5
Annuities.								
Immediate . .	...	...	...	...	2	28	...	

The above figures are based on Returns deposited, for the most part, during the past five years, a are, therefore, merely an approximation to the amount of contracts in force at the present time. In t case of three Companies, namely, the Co-operative, the Customs Fund, and the Northern, the amoun business at a more recent date has been included. The figures of the Colonial and Foreign Companies h been excluded, as their Returns do not separately show the extent of business in the United Kingdom.

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